Photon detectors: Thermal background noise-equivalent power with frequency-dependent detector efficiency

Aditya Raghavan & Sudhanshu S Jha
Department of Physics, Indian Institute of Technology Bombay, Powai, Mumbai 400 076
Received 11 May 2004; accepted 16 August 2004

Background thermal noise often becomes the limiting factor while detecting low signals of photon flux from a distant thermal source. This cannot be eliminated by simply cooling the photodetector. A general method is developed here to calculate the background thermal noise and the noise equivalent power (NEP) for background source temperature $T$, with any given frequency-dependent quantum efficiency function $\eta(\nu)$ for the photodetector. Applications of our analysis to some specific model forms of $\eta(\nu)$, with finite bandwidths, show that earlier calculations are highly inadequate for peak efficiency frequencies $\nu_o < k_BT/h$, where $k_B$ is the Boltzmann constant and $h$ is the Planck constant. The NEP, which determines the limit on the lowest signal power that can be detected by any given photodetector, is much higher in these frequency regions compared to earlier approximate estimates.

[Keywords: Noise-equivalent-power, Thermal background noise, Photon detection]
IPC Code: H 01J 40/00

1 Introduction

The sensitivities of different photon detectors over various range of frequencies and photon counting techniques have made phenomenal progress during the last four decades$^{1,2}$. It is now possible to detect extremely low signals and single photons not only in the infrared and visible regions, but also in other frequency regions using novel semiconductor devices, intricate photo-multiplier systems, superconducting tunnel junctions$^{3,4}$ and other devices. If the detector is cooled to sufficiently low temperatures, the limiting factor for photon detection is usually only the temperature independent quantum photon-noise contribution, unless the photon signal being detected is arising from a thermal source and the corresponding background thermal noise dominates the root means square (rms) fluctuation of the electric current produced by photons. While comparing different photon detectors, a convenient quantity is noise-equivalent power (NEP), which is defined as the equivalent signal power, which will produce a detector output current (or voltage) equal to the rms noise current. It sets the limit for the signal power below which it cannot be detected. For the quantum photon-noise limited detector$^{1,5}$, NEP for photons of energy $h\nu$ is of the order of $h\nu B/\eta$, where $B$ is the bandwidth and $\eta$ is the quantum efficiency of the detector at that frequency. The exact numerical factor, usually between 2 and 4, depends on the specific process involved in converting photons to charge carriers (electrons, etc.) and the resulting current in a given type of detector. In the visible region ($\nu \sim 5 \times 10^{14}$ Hz), with 1Hz bandwidth (which is always used while quoting NEP), the quantum photon noise limited NEP corresponds to about $10^{-18}$ W for unit quantum efficiency. It varies linearly with frequency $\nu$.

For detecting photons from a thermal source it is not possible to eliminate the background noise associated with the source by cooling the detector. In such a situation, the signal photon flux is accompanied by the thermal fluctuation in its flux. The background thermal fluctuation in photon flux or equivalently, thermal fluctuation in the photon energy flux gives rise to NEP, which may be much higher than other sources of noise, including quantum photon-noise. The numerical analysis of NEP due to the background thermal noise has been made earlier by many researchers$^{6,7}$. However, they have mostly made the simplifying approximation in which it is assumed that the frequency-dependent efficiency $\eta(\nu)$ is either a $\delta$-function at each frequency$^5$ or is unity$^6$ for all $\nu$ greater than a low-frequency cut-off $\nu_c$. Limperis$^6$ derived an approximate analytic expression
for the thermal background noise current in the approximation in which it was assumed that $\beta h\nu >> 1$, where $\beta = 1/k_BT$, $k_B$ being the Boltzmann constant and $T$ the background source temperature. This approximate expression has been used by Keyes and Quist for comparing different sources of noise current for a photon detector. However, it must be emphasized that such an expression will no longer be valid unless the temperature $T$ is much lower than $h\nu/c/kB$, where $\nu_c$ is the lowest frequency for which the detector is sensitive. Since the efficiency of any typical photodetector varies with frequency, even if it is limited to a certain finite frequency region of the spectrum, $h\nu/k_BT$ may not be always much greater than unity for the whole frequency region being detected. Thus, the above approach seems to be quite restrictive. Our aim, in this paper, is to remove this restrictive approximation and examine the more general expressions for the background thermal fluctuations in (i) the number-flux $\langle \Delta N^2 \rangle / V_c$, where $V$ is the volume, $c$ is the velocity of light and $\Delta N^2$ is the photon-number dispersion $\langle N^2 \rangle - \langle N \rangle^2$, (ii) the energy-flux $\langle \Delta E^2 \rangle / V_c$, and (iii) the corresponding NEP, with any given frequency dependent efficiency function, $\eta(\nu)$, of the detector.

We first obtain the general expressions for the photon-flux fluctuation, the energy-flux fluctuations and NEP due to thermal background noise. To compare our formulation of the problem with existing approximate analysis, we calculate these quantities numerically for specific forms of $\eta(\nu)$, including Gaussian and rectangular forms (with lower and upper frequency cut-offs).

2 Theory

2.1 Mathematical Formulation of the Problem

The thermodynamic potential $\Omega (T, V, \mu)$ in the grand canonical ensemble of any system of same type of particles is related to the corresponding partition function $Z_{G.C}$ as:

$$\Omega (T, V, \mu) = -(1/\beta) \ln Z_{G.C}, \quad \beta = 1/k_BT \quad \ldots (1)$$

where $\mu$ is the chemical potential and $T$ is the temperature. The average number of particles and the mean square fluctuation (dispersion) in the number are determined by the relations\(^5\):

$$\langle N \rangle = -(\partial \Omega / \partial \mu)_{T, V} \quad \ldots (2)$$

$$\langle \Delta N^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 = -(1/\beta)(\partial^2 \Omega / \partial \mu^2)_{T, V} \quad \ldots (3)$$

For an ideal Bose gas with single-particle energies $\varepsilon_i$, one has:

$$\Omega = \sum_i \Omega_i, \quad \Omega_i = (1/\beta) \ln [1 - \exp(\beta(\varepsilon_i - \mu))] \quad \ldots (4)$$

$$\langle N \rangle = \sum_i \langle N_i \rangle, \quad \langle N_i \rangle = \left[ \exp(\beta(\varepsilon_i - \mu)) - 1 \right]^{-1} \quad \ldots (5)$$

$$\langle \Delta N^2 \rangle = \sum_i \langle \Delta N_i^2 \rangle, \quad \langle \Delta N_i^2 \rangle = \left[ \exp(\beta(\varepsilon_i - \mu)) \right] \langle N_i \rangle^2 \quad \ldots (6)$$

Going over to the continuous frequency spectrum for photons with $\mu = 0$, in a very large volume $V$, with $\varepsilon = h\nu$, one obtains

$$\langle N \rangle / V = \langle n \rangle = \int_0^\infty dv n_v, \quad n_v = (8\pi \nu^2/c^3) \left[ \exp(\beta h\nu) - 1 \right]^{-1} \ldots (7)$$

$$\langle \Delta N^2 \rangle / V = \int_0^\infty dv \langle \Delta N^2 \rangle_v / V, \quad \langle \Delta N^2 \rangle_v / V \quad \ldots (8)$$

$$\langle \Delta E^2 \rangle / V = \int_0^\infty dv \langle \Delta E^2 \rangle_v / V, \quad \langle \Delta E^2 \rangle_v / V = \left[ \langle \Delta N^2 \rangle_v / V \right] h^2 \nu^2 \quad \ldots (9)$$

Consider now a typical photodetector placed in a heat-sink inside a cavity with an opening specified by the full cone acceptance angle $\theta$ (see Fig. 1), which is

![Fig. 1 — Sketch of the detector geometry placed in a heat sink inside a cavity with an opening defined by full cone angle $\theta$, exposed to photons from a thermal source at background temperature $T$](image-url)
detecting photons from a thermal source at the background temperature $T$ being emitted in all directions. The elementary kinetic theory tells us that the maximum number of photons per unit time, which can strike any surface of unit area in a given direction from the source is $(c/4)\langle N \rangle/V$. If the dimensionless quantum efficiency of the cooled photodetector is $\eta$ for detecting photons of frequency $\nu$ and if $G$ is the dimensionless gain factor in the detector system per electronic charge carrier produced by the photons, for a signal power $P_s(\nu)$ for photons of frequency $\nu$ in the range $\Delta\nu$ the resulting signal current is:

$$i_s(\nu) = \left[ P_s(\nu)/h\nu \right] Gen$$

where $e$ is the magnitude of the electronic charge. For the background thermal noise (see Fig. 1), the total mean square noise current in the range $\Delta\nu$ can be written as$^6,7$:

$$\langle \Delta_i^2 \rangle = 4G^2e^2\left(c/4V\right)\langle \Delta N^2 \rangle \eta A \Delta \nu \sin^2(\theta/2) \quad \cdots(11)$$

where $A$ is the detector area. It should be emphasized here that the factor 4 in front of the expression given in Eq. (11) and the factor $(1/4)$ in flux striking the detector are not universal. They depend on the exact physical process involved in detecting photons and the geometrical factors respectively. However, here these factors cancel each other.

The noise equivalent power, $P_{\text{NEP}}(\nu)$ in W, is defined as the equivalent signal power which gives

$$\langle \Delta_i^2 \rangle = i_s^2(\nu),$$

i.e.,

$$P_{\text{NEP}}(\nu) = \left[ \left\langle \Delta_i^2 \right\rangle h^2\nu^2 \right]^{1/2}/Gen = \left[ \left\langle \left( c\langle \Delta E^2 \rangle \varepsilon^2 \right)/V \right\rangle \eta \Delta \nu \right]^{1/2} (\Delta \nu)^{1/2} A^{1/2} \sin(\theta/2)/\eta \quad \cdots(12)$$

Following the usual convention of quoting $P_{\text{NEP}}$ for detector area $A = 1$ cm$^2$, frequency range $\Delta\nu = 1$ Hz, $\eta = 1$ and the full cone acceptance angle $\theta = \pi$, we show in Fig. (2) a plot of $P_{\text{NEP}}$ in W, using Eqs (8) and (9) for $\left\langle \Delta E^2 \right\rangle/V$. This is the standard and most commonly used plot$^1,5$ in analyzing the thermal background noise.

For the more general case in which the quantum efficiency $\eta$ and the response of the detector depends on $\nu$, we must generalize the expression given in

$$\eta(\nu) = \eta_0 \left\langle B \right\rangle_D f_\nu, \quad \int_0^\infty f_\nu d\nu = 1$$

$$\left\langle B \right\rangle_D = \int_0^\infty d\nu (\eta(\nu)/\eta_0), \quad \eta_0 = \left\langle 1/\left\langle B \right\rangle_D \right\rangle \int_0^\infty \eta(\nu) d\nu$$

where $\eta_0$ is the peak (maximum) efficiency of the detector. Note that we have introduced over here a normalized efficiency distribution function $f_\nu$ with the dimension of Hz$^{-1}$ and we have defined an effective frequency-bandwidth $\left\langle B \right\rangle_D$ of the detector through
Eq. (14). This is not necessarily equal to the usual 3-dB frequency bandwidth used more commonly.

The total $P_{\text{NEP}}$ integrated over the full detector response is given by:

$$P_{\text{NEP}} = \left( \int dv \left[ c \frac{\langle \Delta E^2 \rangle_{v'} / V}{\langle 1/b \rangle} \eta(v) \right]^{1/2} \right)^{1/2}$$

$$\langle B \rangle_D^{1/2} A^{1/2} \sin(\theta/2) \left[ \frac{1}{\langle 1/b \rangle} \right]^{1/2} \int_0^\infty dv \eta(v)$$

$$\cdots (15)$$

While arriving at the relation given in Eq. (15), we have essentially replaced $\eta$ in the denominator of Eq. (12), arising from the signal detection, by $\eta$ given by Eq. (14). Using Eqs (13) and (14), the expression given in Eq. (15) can be written as:

$$P_{\text{NEP}} = \left( \int_0^\infty dv \left[ c \frac{\langle \Delta E^2 \rangle_{v'} / V}{\langle 1/b \rangle} \eta(v) \right]^{1/2} \right)^{1/2}$$

$$\langle B \rangle_D^{1/2} \eta_0^{-1/2} \sin(\theta/2)$$

$$\cdots (16)$$

Again, if we follow the usual convention of quoting total $P_{\text{NEP}}$ in watts for $\eta_0 = 1$ and $\theta = \pi$, $P_{\text{NEP}}$ can be simplified to the form:

$$P_{\text{NEP}} = \left( \int_0^\infty dv \left[ 8 \pi \frac{h^2 / c^2}{4} \right]^{1/2} \frac{v^2 \eta(v) \exp(\beta hv)}{ \left[ \exp(\beta hv) - 1 \right]^2} \right)^{1/2}$$

$$\int_0^\infty dv \eta(v) \langle B \rangle_D^{1/2} A^{1/2}$$

$$\cdots (17)$$

in which we have to put $\langle B \rangle_D = 1$ Hz, $A = 1$ cm$^2$. Again, we should emphasize that taking $\eta_0 = 1$ (its maximum possible value) implies that we are quoting a lower bound for $P_{\text{NEP}}$.

Note that if one takes $f_c$ to be a delta function $\delta(v)$ in the general form given in Eq.(13) for $\eta(v)$, the expression given in Eq. (17), with $\langle B \rangle_D = 1$ Hz and $A = 1$ cm$^2$, leads to the exactly same result of $P_{\text{NEP}}$ as plotted in Fig. (2) using the form given in Eq. (12), with $A = 1$ cm$^2$, $\Delta v = 1$ Hz, $\eta = 1$ and $\theta = \pi$. Our general expression given in Eq. (17), with $\langle B \rangle_D = 1$ Hz and $A = 1$ cm$^2$, can be used to calculate $P_{\text{NEP}}$ in watts for any given distribution of the detector efficiency $\eta(v)$ which can be measured separately.

We can follow the above procedure to write down the more general expressions for the photon flux, $(c/4)\langle N / V \rangle$ and for the mean square photon fluctuation flux $(c/4)\langle \Delta N^2 / V \rangle$, falling on the detector with a given efficiency distribution function $\eta(v)$. We find:

$$\left( \frac{c}{4} \right) \langle N / V \rangle = \left\{ \left( \frac{2\pi}{c^2} \right) \int_0^\infty dv \frac{v^2 \eta(v)}{\exp(\beta hv) - 1} \right\}$$

$$\int_0^\infty dv \eta(v) \langle B \rangle_D \cdots (18)$$

$$\left( \frac{c}{4} \right) \langle \Delta N^2 / V \rangle = \left\{ \left( \frac{2\pi}{c^2} \right) \int_0^\infty dv \frac{v^2 \eta(v) \exp(\beta hv)}{\left[ \exp(\beta hv) - 1 \right]^2} \right\}$$

$$\int_0^\infty dv \eta(v) \langle B \rangle_D \cdots (19)$$

3 Results

3.1 Model Calculations and Discussion of Results

In the previous section, we have presented a general formulation of the problem of calculating background thermal noise from a thermal source of photons, as seen by a photodetector with any given frequency distribution $\eta(v)$ for its quantum efficiency. As emphasized earlier, the functional form of $\eta(v)$ depends, of course, on the type of photodetector being used. This has to be obtained separately by proper measurement. Nevertheless, it is very instructive to apply our general formulation to a few specific models for $\eta(v)$.

Form (I):

$$\eta(v) = 0 \ , \ v < v_c$$

$$\eta(v) = 1 \ , \ v \geq v_c$$

$$\cdots (20)$$
where \( \eta_0 = 1 \) and by definition,
\[
\frac{1}{\langle B_D \rangle} \int_0^\infty d\nu \eta(\nu) = 1 \tag{21}
\]

**Form (II):**
\[
\eta(\nu) = \begin{cases} 
0 & , \nu < \nu_L \\
\eta_0 & , \nu_L \leq \nu \leq \nu_U \\
0 & , \nu > \nu_U 
\end{cases} \tag{22}
\]

where \( \nu_L \) and \( \nu_U \) are the lower and upper cut-off frequencies for a rectangular efficiency function, and where by definition the equivalent bandwidth is given by:
\[
\langle B_D \rangle = \int_0^\infty d\nu \frac{\eta(\nu)}{\eta_0} = \nu_U - \nu_L \tag{23}
\]

**Form (III):**
\[
\eta(\nu) = \eta_0 \exp \left[ -\left( \frac{\nu - \nu_0}{4B^2 / \pi} \right)^2 \right] \equiv \eta_0 \exp \left[ -\left( 1 - \nu_0 \right)^2 / 2a^2 \right] \tag{24}
\]

where \( a = \left( \frac{2}{\pi} \right)^{1/2} B/\nu_0 \) \tag{25}

In this case, the effective detector bandwidth \( \langle B_D \rangle \) is given by:
\[
\langle B_D \rangle = \int_0^\infty d\nu \exp \left[ -\left( \nu - \nu_0 \right)^2 / 4B^2 / \pi \right] \tag{26}
\]

Note that for \( \nu_0 = 0 \), we have \( \langle B_D \rangle = B \).

The forms of \( \eta(\nu) \) in the above cases are sketched in Figs (3 and 4).

The simple form (I), Eq. (20), with a simple lower cut-off was used by Limperis, who obtained an analytic expression for the integrated value of the

---

**Fig. 3** — Sketch of the model efficiency \( \eta(\nu) \) used by Limperis, defined as the form (I) in the text.

**Fig. 4** — Forms (II) and (III) (see text) for the efficiency function \( \eta(\nu) \) also used here.

**Fig. 5** — Plot showing our results for the total mean square number fluctuation flux \( \langle c/4 \rangle \langle \Delta N^2 \rangle > / V \) as seen by the photodetector, for the form (I) (see text). The total mean square number fluctuation Flux obtained by Limperis is also plotted here for comparison. The classical Maxwell-Boltzmann value \( \langle c/4 \rangle \langle N \rangle > / V \) is also plotted to show how it differs from the case of photons.
flux \((c/4)\langle\Delta N^2\rangle/V\), as given by Eq. (19), in the approximation in which \(\beta \nu > k_B T/h\). Here, we integrate Eqs (18) and (19) numerically, with the form (I), without making the approximation of Limperis. Our results for \((c/4)\langle\Delta N^2\rangle/V\) and \((c/4)\langle N\rangle/V\) are plotted in Fig. (5). We see that the exact result differs appreciably in the region \(\nu < k_B T/h\) from the approximate calculation of Limperis, displayed on the same plot. We also find that for \(\beta \nu > k_B T/h\), in any case the flux of mean square fluctuation approaches the classical value \((c/4)\langle N\rangle/V\), as expected from the classical Maxwell-Boltzmann statistics. In other words, the analysis of Limperis is inadequate to take into account the Bose statistics obeyed by photons.

For the form (III) for \(\eta(\nu)\), we consider different values of the dimensionless quantity \(\beta \nu_0\), i.e. different values of the frequency \(\nu_0\) (in units of \(k_B T/h\)) at which the quantum efficiency has its maximum value \(\eta_0\), and different values of the bandwidth parameter \(a = (2/\pi)^{1/2} B / \nu_0\). We find that for the same values of the effective bandwidth \(\langle B \rangle_D\) defined by Eq. (25) for the form (III) of \(\eta(\nu)\) and by Eq. (23) for

![Fig. 6](https://via.placeholder.com/150)

**Fig. 6** — Plot of the root mean square fluctuation in photon flux \((c/4)\langle\Delta N^2\rangle/V\) and the square root of the average flux \((c/4)\langle N\rangle/V\) of photons as seen by the detector, for different values of the effective bandwidth and the peak-efficiency frequency \(\nu_0\) of the detector with \(\eta(\nu)\) of the form (III) (see text).

![Fig. 7](https://via.placeholder.com/150)

**Fig. 7** — For the efficiency function of the form (III) (see text), a plot of the Noise-Equivalent Power \((P_{\text{NEP}})\) in Watts for different values of the peak-efficiency frequency \(\nu_0\) and the bandwidth parameter \(a\). The plot is for the detector area, \(A = 1\) cm², effective bandwidth, \(\langle B \rangle_D = 1\) Hz, \(\theta = \pi\) and \(\eta_0 = 1\).

![Fig. 8](https://via.placeholder.com/150)

**Fig. 8** — Plot of the root mean square fluctuation of the photons and the root mean flux of the photons arising from the universal cosmic microwave radiation (CMR) at \(T = 2.73\) K. Here, a typical bandwidth parameter \(a = 2.5\), i.e., \(\langle B \rangle_D = 2.5\) \(\nu_0(\pi/2)^{1/2}\) is assumed. The plot is as a function of the peak detector efficiency frequency \(\nu_0\) in the units of \(k_B T/h = 5.7 \times 10^{10}\) Hz.
the form (II), and same $\eta_0$, our numerical results for the photon flux at the detector, the flux of mean square photon number fluctuation and $P_{\text{NIP}}$ are extremely similar. We, therefore, present here our numerical results in Figs (6, 7 and 8), only for the form (III) of $\eta(v)$.

In Fig. 6, we show how the root-mean square fluctuation in photon flux $\sqrt{\langle (c/4) \langle N^2 \rangle / \nu \rangle^{1/2}}$ differs from the square-root of the average flux of photons $\sqrt{\langle (c/4) \langle N \rangle / \nu \rangle^{1/2}}$ as seen by the detector, for different values of the effective bandwidth and the peak-efficiency frequency $\nu_0$ of the detector. They differ considerably for low values of $\nu_0$ in units of $k_B T / h$. In Fig. 7, we plot $P_{\text{NIP}}$ in watts for different values of the peak-efficiency frequency $\nu_0$ and the effective bandwidth parameter, $a$, of the detector, for detector area $A = 1 \text{cm}^2$, $B_D = 1 \text{ Hz, } \theta = \pi$ and $\eta_0 = 1$. As the bandwidth parameter $a \equiv (2 / \pi)^{1/2} B / \nu_0$ goes to zero, $P_{\text{NIP}}$ can be shown to approach the widely used curve plotted in Fig. (2), as it should. However, for practical values of $a$, $P_{\text{NIP}}$ is much higher for peak-efficiency frequencies $\nu_0 \leq (h/k_B T)$, whereas usual $T$ is the thermal source temperature.

For the universal cosmic microwave radiation (CMR) at temperature $T = 2.73 \text{ K}$, we have plotted in Fig. 8, the root-mean square flux of photons, as seen by the detector for a typical bandwidth parameter $a = 2.5$, i.e., $B_D = 2.5 \nu_0 (\pi/2)^{1/2}$, as a function of the peak-efficiency frequency $\nu_0$. We again find that the deviation of fluctuation is large compared to the classical case (where both curves should coincide), for $\nu_0 < k_B T / h = 5.7 \times 10^{10} \text{ Hz}$. For measuring small anisotropies in CMR in different directions, the analysis must take into account the effect of the background thermal noise detected by the photodetector (with a given efficiency distribution $\eta(v)$) along with the signal.

In conclusion, we will like to emphasize that while detecting photons from a thermal source, a proper analysis of the root-mean square fluctuation in photon number flux and of $P_{\text{NIP}}$ must be made taking into account the measured frequency-dependent quantum efficiency function $\eta(v)$ of the detector. We have presented here a convenient general method to make such an analysis.

Acknowledgement

One of the authors (SSJ) would like to thank the Department of Atomic Energy for the fellowship awarded to him as DAE-BRNS Senior Scientist. He will also like to thank IIT Bombay for all the facilities provided to him to work here, and to the Tata Institute of Fundamental Research, Mumbai, for the Honorary Professorship there. The first author (AR) wishes to thank Swapnil Jawkar of IIT Bombay for his assistance in this work.

References