Thermal post-buckling behaviour of tapered columns—A simple solution

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The applicability of a simple method proposed earlier by the authors to predict the thermal post-buckling behaviour of structural elements has been demonstrated in this paper for the case of tapered columns. Numerical results are presented to show the accuracy of the method along with comparable finite element results. Both depth and diameter tapered columns are considered with simply supported and clamped boundary conditions.

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Dym1 and Thompson and Hunt2 have discussed the post-buckling behaviour of uniform columns subjected to mechanical loads. The ends of columns are allowed to move axially and the non-linear moment curvature relations are considered in this case. However, when the columns are subjected to a uniform temperature rise, an axial compressive load is generated when the ends of the column are considered axially immovable. The non-linearity in this case is in the strain-displacement relations. Using the versatile finite element method3 the authors have investigated the thermal post-buckling of uniform columns, when the column is subjected to a uniform temperature rise from the stress free state and the ends are axially immovable. The post-buckling of tapered columns is also studied4,5 using the versatile finite element and the well known Rayleigh-Ritz methods.

The tapering columns are widely used in aerospace and many engineering structures as they offer cost effective and near optimum configurations (in the sense of higher critical loads for the same volume of material as compared to the uniform columns). Further tapered columns with circular cross-section have the advantage of buckling at the same critical load (for both mechanical and thermal loads) in any bending plane.

The applicability of a simple method proposed earlier by the authors6, which provided very accurate results in their preliminary study of thermal post-buckling uniform columns, is further demonstrated in this paper for the thermal post-buckling behaviour of tapered columns. As discussed earlier6, this method requires the knowledge of the linear thermal buckling load parameter and the tension (parameter) developed in the columns because of large deformation, to predict the thermal post-buckling load. Further, an assumption of the axial displacement distribution, which is essential in the finite element and Rayleigh-Ritz formulations, which increases the number of degrees of freedom in the finite element method and makes it a difficult task for some boundary conditions in the case of the Rayleigh-Ritz method, is not necessary in the present method.

Present Method

Based on the physics of the problem, the simple method of predicting the thermal post-buckling behaviour of tapered columns involves two steps. The first step is the evaluation of linear thermal load parameter \( \lambda_c = \frac{(P)_c L^2}{EI_c} \) . This step can be easily carried out using any of the methods like Rayleigh-Ritz or finite element method. Any accurate linear thermal load parameter available otherwise also can be considered. For completeness a Rayleigh-Ritz solution with two-term displacement distribution is given in Appendix for both simply supported and clamped tapered beams. Depth taper as well as the diameter taper is considered. It may be noted that the linear results obtained with two-term displacement distribution are in close agreement with the finite element results.

To predict the thermal post-buckling behaviour of the columns, the next step is to evaluate the tension developed in the columns due to large deformations. With the physics of the problem considered, the
column buckles at $\lambda_L$ values and any increase in temperature (or thermal load) causes lateral deformations. A finite lateral displacement $a$ at the center of the column generates an axial tension, represented by a thermal equivalent axial tension parameters $\lambda_{T_1}$. Thus for any given lateral displacement, the column can take a further temperature rise, which induces the additional compressive thermal equivalent load, which is $\lambda_{T_1}$.

The post-buckling thermal equivalent load parameter $\lambda_{NL}$ can thus be written as:

$$\lambda_{NL} = \lambda_L + \lambda_{T_1} \quad \text{... (1)}$$

or

$$\frac{\lambda_{NL}}{\lambda_L} = 1 + \frac{\lambda_{T_1}}{\lambda_L} \quad \text{... (2)}$$

The tension $T_a$ can be obtained following Woinowsky-Krieger for tapered columns as:

$$T_a = \frac{EI_x}{2r^2} \int_{-l}^{l} w^2 \, dx/I_n \quad \text{... (3)}$$

where

$$I_n = \int_{-l}^{l} \frac{dx}{F(x)} \quad \text{... (4)}$$

and $F(x)$ are the variations of area of cross-section given in (A.3) and (A.5) of Appendix for depth and diameter tapers respectively. The non-dimensional tension parameter $\lambda_{T_1} (= T_a L^2 / EI_x)$ can be obtained after substitution of the linear mode shape for $w$, evaluated corresponding to $\lambda_L$ values in terms of the amplitude, and also ensuring that the maximum amplitude of $b_1$ is at the center of the column. With $\lambda_L$ and $\lambda_{T_1}$ available the ratios of $\lambda_{NL} / \lambda_L$ can be easily evaluated for different values of the ratio of central amplitude, $b_1$ to radius of gyration at the center of the column, $r_c$. In the above, $P_L$, $P_{NL}$ and $T_a$ can all be expressed in terms of appropriate temperature rise, $T$, as $2EA_L\alpha T/I_n$, $\alpha$ being the coefficient of thermal expansion.

**Results and Discussion**

Numerical results are presented for both simply supported and clamped columns of depth and diameter tapers, based on the earlier presented formulations. The results in terms of $\lambda_L$ and a coefficient $\bar{c}$, defined as:

$$\frac{\lambda_{NL}}{\lambda_L} = 1 + \bar{c} \left( \frac{h}{r_c} \right)^2 \quad \text{... (6)}$$

are presented for different values of the taper parameter $\beta$. Corresponding finite element results are also included for comparison.

Table 1 gives the $\lambda_L$ and $\bar{c}$ values for simply supported depth and circular taper columns (with appropriate definition of $\beta$). Table 2 gives the results for clamped columns. The $\lambda_L$ values are obtained using the Rayleigh-Ritz scheme given in the Appendix. It can be seen from the tables that the present results are in good agreement with those from the finite element method. However, it may be noted that $\bar{c}$ values of finite element method are obtained based on a least square fit from the $\lambda_{NL} / \lambda_L$ values for different $\left( \frac{h}{r_c} \right)$ values. The slight differences in the

| Taper Parameter $\beta$ | Depth Taper | F.E. | | Diameter Taper | F.E. |
|-------------------------|------------|------|----------------|------|
| $\lambda_L$ | $\bar{c}$ | $\lambda_L$ | $\bar{c}$ | $\lambda_L$ | $\bar{c}$ |
| 0.0 | 2.4674 | 0.2500 | 2.4674 | 0.2500 | 2.4674 | 0.2500 |
| 0.1 | 2.3670 | 0.2651 | 2.3667 | 0.2653 | 2.4165 | 0.2613 |
| 0.2 | 2.2580 | 0.2838 | 2.2573 | 0.2842 | 2.3554 | 0.2762 |
| 0.3 | 2.1390 | 0.3076 | 2.1373 | 0.3080 | 2.2828 | 0.2959 |
| 0.4 | 2.0096 | 0.3379 | 2.0038 | 0.3396 | 2.2017 | 0.3207 |
| 0.5 | 1.8716 | 0.3764 | 1.8527 | 0.3838 | 2.1229 | 0.3492 |
values for higher $\beta$-values, could be due to the approximation function chosen for $w$ in the present method. However, by including additional terms in the displacement distributions, the accuracy will improve, as it was observed that there was an enormous improvement between one term solution to two term solution.

Conclusions
A simple method is presented in this paper for evaluating the post-buckling behaviour of tapered columns. It is demonstrated that the method, considering the physics of the problem, yields accurate results, provided appropriate displacement distributions are assumed. Numerical results are presented for depth and diameter taper columns with simply supported and clamped boundary condition and varying taper parameter values. Linear thermal load parameters $\lambda_\text{L}$ and a coefficient, $\gamma$ of the ratio of non-linear to linear thermal load parameters are given in the tables. This method can be easily extended to any other configurations of structural elements and involves very little computational times.

Appendix
The first step of the present method is the evaluation of linear thermal load parameter $\lambda_\text{L} = (P_L L^2 / EI)$ by considering the linear strain energy, $U$ and the work done, $W$ by the thermal load in terms of lateral displacement, $w$ given by:

$$U = \frac{E}{2} \int_{-L}^{L} w'^2 \, dx$$  \quad \cdots (A.1)$$

and

$$W = \frac{P}{2} \int_{-L}^{L} w^2 \, dx$$  \quad \cdots (A.2)$$

where $2L$ is the length of the column, $E$ is the Young’s modulus and $P$ is the mechanical equivalent of applied thermal load in terms of temperature rise. Subscript ‘cr’ indicates the critical value. The area of cross-section, $A$ and the area moment of inertia, $I$ have been considered to have the following variations in terms of axial coordinate, $x$ for the depth and diameter taper columns (Fig. 1):

Depth taper

$$A = A_c \left(1-\frac{\beta x}{L}\right) \text{ for } 0 \leq x \leq L$$

$$= A_c \left(1+\frac{\beta x}{L}\right) \text{ for } -L \leq x \leq 0$$  \quad \cdots (A.3)$$

$$I = I_c \left(1-\frac{\beta x}{L}\right)^3 \text{ for } 0 \leq x \leq L$$

$$= I_c \left(1+\frac{\beta x}{L}\right)^3 \text{ for } -L \leq x \leq 0$$  \quad \cdots (A.4)$$

Diameter taper

$$A = A_c \left(1-\left(\frac{\beta x}{L}\right)^2\right) \text{ for } 0 \leq x \leq L$$

$$= A_c \left(1+\left(\frac{\beta x}{L}\right)^2\right) \text{ for } -L \leq x \leq 0$$  \quad \cdots (A.5)$$

References
\[ I = I_c \left(1 - \frac{\beta x}{L}\right)^4 \quad \text{for} \ 0 \leq x \leq L \]
\[ = I_c \left(1 + \frac{\beta x}{L}\right)^4 \quad \text{for} \ -L \leq x \leq 0 \quad \ldots \ (A.6) \]

\( \beta \) is the taper parameter defined in terms of depth \( d \) and diameter \( D \) as:
\[ \beta = \frac{d_c - d_e}{d_e} \quad \text{for depth taper} \]
and
\[ \beta = \frac{D_c - D_e}{D_e} \quad \text{for diameter taper} \]

Subscripts \( c \) and \( e \) indicate the depth and diameter are at center and end of the column respectively.

A two term displacement distribution for \( w \), which approximates the buckled shape of the tapered column in an accurate way and satisfies the boundary conditions is assumed as:
\[ w = h_1 \cos \frac{\pi x}{2L} + h_2 \cos \frac{3\pi x}{2L} \quad \ldots \ (A.7) \]

for simply supported boundary conditions. For clamped conditions accurate algebraic expressions satisfying the boundary conditions have been chosen as:
\[ w = \left[1 + \frac{a_1}{4} \left(\frac{x}{L}\right)^2 + \frac{a_3}{16} \left(\frac{x}{L}\right)^4\right] + \left[\frac{\alpha_1}{4} \left(\frac{x}{L}\right)^2 + \frac{\alpha_4}{16} \left(\frac{x}{L}\right)^4 + \frac{1}{64} \left(\frac{x}{L}\right)^6\right] \quad \ldots \ (A.8) \]

### Table 3—Coefficients of characteristic Eq (A.9) for simply supported columns

<table>
<thead>
<tr>
<th>Depth taper column</th>
<th>Diameter taper column</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a = 1 )</td>
<td>( a = 1 )</td>
</tr>
<tr>
<td>( b = -\frac{\pi^2}{2} [A_1 + 9A_4] )</td>
<td>( b = -\frac{\pi^2}{2} [A_1 + 9A_4] )</td>
</tr>
<tr>
<td>( c = \frac{9}{4} \pi^2 [A_1 - \pi^2 A_4] )</td>
<td>( c = \frac{9}{4} \pi^2 [A_1 - A_4^2] )</td>
</tr>
<tr>
<td>( A_1 = \frac{1}{2} - 3\beta \left(\frac{1}{4} - \frac{1}{\pi^2}\right) + 3\beta^2 \left(\frac{1}{6} - \frac{1}{\pi^2}\right) - \beta^3 \left(\frac{1}{8} - \frac{3}{2\pi^2} + \frac{6}{\pi^2}\right) )</td>
<td>( A_1 = \frac{1}{2} - 3\beta \left(\frac{1}{4} - \frac{1}{\pi^2}\right) + 3\beta^2 \left(\frac{1}{6} - \frac{1}{9\pi^2}\right) - \beta^3 \left(\frac{1}{8} - \frac{1}{6\pi^2} + \frac{6}{81\pi^2}\right) )</td>
</tr>
<tr>
<td>( A_4 = \frac{\beta^3}{\pi^2} - \frac{9\beta^2}{4\pi^2} - \beta \left(\frac{6}{\pi^2} - \frac{9}{8\pi^2}\right) )</td>
<td>( A_4 = \frac{\beta^3}{\pi^2} - \frac{9\beta^2}{2\pi^2} - \beta \left(\frac{6}{\pi^2} - \frac{9}{8\pi^2}\right) + \beta \left(\frac{45}{4\pi^4} - \frac{3}{2\pi^2}\right) )</td>
</tr>
<tr>
<td>( A_4 = \frac{4\beta^3}{\pi^2} - \frac{9\beta^2}{2\pi^2} + \beta \left(\frac{6}{\pi^2} - \frac{9}{8\pi^2}\right) )</td>
<td>( A_4 = \frac{4\beta^3}{\pi^2} - \frac{9\beta^2}{2\pi^2} + \beta \left(\frac{6}{\pi^2} - \frac{9}{8\pi^2}\right) + \beta \left(\frac{45}{4\pi^4} - \frac{3}{2\pi^2}\right) )</td>
</tr>
</tbody>
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Fig. 1—Geometry of tapered column

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RAJU & RAO: THERMAL POST-BUCKLING BEHAVIOUR OF TAPERED COLUMNS
Depth taper column

\[ a = A_1 A_1 \]
\[ b = -[A_2 A_2 + A_2 A_2] \]
\[ c = A_1 A_1 + A_1 \]

\[ A_1 = \frac{64}{5} - 24\beta^2 + \frac{704}{35}\beta^2 - 6\beta^4 \]
\[ A_2 = \frac{128}{105} \]
\[ A_3 = \frac{1}{35} - \frac{9\beta}{64} + \frac{\beta^2}{7} - \frac{3\beta^2}{64} \]
\[ A_4 = \frac{3}{2}(\frac{3\beta}{2240} - \frac{24\beta^2}{1024} - \frac{3\beta^2}{24640} - \frac{3\beta^3}{4096}) \]
\[ A_5 = \frac{1}{36960} \]

with \( \alpha_1 = -8, \alpha_2 = 16, \alpha_3 = -1/16 \) and \( \alpha_4 = -1/2 \) and \( b_1, b_2 \) are the undetermined coefficients.

The total potential energy \( \Pi = U - W \) can be computed after substituting appropriate expressions from (A.3) to (A.6) and with (A.7) or (A.8) and carrying out the integrations. Minimization of the total potential energy with respect to \( b_1 \) and \( b_2 \) yields the characteristic equation, involving the linear thermal load parameter, \( \lambda_L \), as a quadratic equation of the form:

\[ a \lambda_L^2 + b \lambda_L + c = 0 \]  

… (A.9)

Expressions for \( a, b \) and \( c \) for different types of taper and boundary conditions are given in Tables 3 and 4. Eq (A.9) can be solved to obtain the linear buckling loads for different cases.

<table>
<thead>
<tr>
<th>Diameter taper column</th>
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</thead>
<tbody>
<tr>
<td>( a = A_1 A_1 )</td>
</tr>
<tr>
<td>( b = -[A_2 A_2 + A_2 A_2] )</td>
</tr>
<tr>
<td>( c = A_1 A_1 + A_1 )</td>
</tr>
<tr>
<td>( A_1 = \frac{64}{5} - 32\beta^2 + \frac{1408}{35}\beta^2 - \frac{192\beta^2}{35} )</td>
</tr>
<tr>
<td>( A_2 = \frac{128}{105} )</td>
</tr>
<tr>
<td>( A_3 = \frac{1}{35} + \frac{3\beta^2}{16} + \frac{\beta^2}{7} - \frac{3\beta^2}{16} + \frac{53\beta^4}{1155} )</td>
</tr>
<tr>
<td>( A_4 = \frac{3}{2240} - \frac{24\beta^2}{1024} - \frac{24\beta^2}{24640} + \frac{643\beta^4}{960960} )</td>
</tr>
<tr>
<td>( A_5 = \frac{1}{36960} )</td>
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</table>