Simulations of thermo-magnetic convection in an annulus between two concentric cylinders†

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In this paper the thermo-magnetic convection of a magnetic fluid held between two vertical concentric cylinders is studied. Three-dimensional computer simulations of thermo-magnetic convection have been conducted under an azimuthal magnetic field, when the inner cylinder is kept at a higher temperature than the peripheral one. Simulations have been carried out both in the absence and in the presence of gravity. In the absence of gravity, the problem will be analogous to the classical Rayleigh-Bernard instability in a radial gravitation field. The objective of this study is to investigate the combined effects of magnetic and buoyancy driven convection. In the absence of gravity the simulations revealed the expected flow pattern consisting of counter rotating convection cells with diameter equal to the thickness of the fluid layer. When the gravity is introduced, the case may become unsteady consisting of cells drifting upward. For small gravitational Rayleigh numbers introduction of magnetic field gradient may stabilize the flow. The onset of instability depends on the magnetic and the gravitational Rayleigh numbers and their ratio, radius and aspect ratios of the cylinders as well as on the fluid properties.

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When a temperature difference is applied to a layer of magnetic fluid under magnetic field gradient, the system is potentially unstable, and with certain conditions a convective flow may set in. In this paper the combined effects of buoyancy and magnetic force on the magnetic fluid convection have been studied. Computer simulations have been conducted in order to study the magnetic fluid convection in a vertical cylindrical annulus, in the presence of radial temperature and magnetic field gradients.

In the absence of magnetic field gradient and in the presence of gravity, the buoyancy driven convection will take place, leading to purely meridional convection. Buoyancy driven convection in a cylindrical annulus has been studied, e.g., by Lee et al.1. In the absence of gravity, the azimuthal convection with counter rotating cells takes place in an annulus, because of the radial temperature and magnetic field gradients, and the temperature dependence of the fluid magnetization. This phenomenon has been studied earlier both by computational methods2 and by experiments under microgravity conditions3.

Most of the classical studies, both in buoyancy driven convection4, and thermo-magnetic convection5 are related to the two-dimensional plane and axisymmetric flows. However, the combination of magnetic convection caused by radial magnetic field gradient and natural convection caused by gravity vector parallel to the cylinder axis leads to complicated three-dimensional flow. Recently, it was shown that the onset of convection may be three-dimensional also in the absence of gravity6. In addition to two-dimensional approximation, two simplifications have often been made in the studies of thermo-magnetic convection, namely, the superposition of parallel or antiparallel gravitational and magnetic body forces or the assumption of a strong magnetic field gradient and negligible gravitational convection. In this study the direction of the gravity and the magnetic field gradient are perpendicular to each other so that these two phenomena cannot be presented with a single effective body force term. Also we will mainly focus at the cases, where the contributions of magnetic and buoyant forces are of the same order of magnitude, and neither one is clearly dominant.

Problem Description and the Governing Equations

Let us consider an annular cavity between two concentric cylinders shown in Fig. 1. Magnetic fluid is held in an annulus and the inner cylinder is kept at a
higher temperature than the peripheral one. An electric current is led through the inner cylinder to produce an azimuthal magnetic field, decreasing in the radial direction. Because of the temperature dependence of the fluid magnetization, a magnetization gradient opposite to the temperature gradient is generated.

When a fluid element with colder temperature and therefore higher magnetization is moved towards the inner cylinder it experiences a higher magnetic force than the warmer fluid surrounding it. This magnetic force is effecting in the direction of magnetic field gradient and tends to move the fluid element further towards the centre of the cylinder leading to instability. The system is similar with the Rayleigh-Bernard convection of an ordinary fluid under a radial gravitational field.

The equations governing the magnetic fluid motions are continuity and momentum equations with thermal energy balance. For small temperature and magnetic field differences the thermodynamic and transport properties of magnetic fluid can be considered constant, except for the density in the gravitational term and the magnetization in the magnetic force term, which are allowed to vary with temperature and latter also with magnetic field magnitude to generate the buoyancy and magnetic force. If the fluid is further assumed incompressible, the governing equations for magnetic fluid may be written as

\[ \nabla \cdot \mathbf{u} = 0 \quad \cdots (1) \]
\[ \rho_0 \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \rho(T) \mathbf{g} + \mu \nabla^2 \mathbf{u} \cdots (2) \]
\[ + \mu_0 M(T,H) \nabla H + \mathbf{B} \times (\nabla \times \mathbf{h}) \]
\[ \left[ \rho c_{v,H} - \mu_0 H \left( \frac{\partial M}{\partial T} \right)_{v,H} \right] \left( \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) \cdots (3) \]
\[ + \mu_0 \left( \frac{\partial M}{\partial T} \right)_{v,H} \frac{dH}{dt} = k \nabla^2 T + \mu \Phi \]

Last term in the momentum equation (2) represents dissipative or off-equilibrium magnetic force, which may be neglected for stationary applied field and small velocities considered in free convection flows. For stationary field also the last term on the left hand side of energy equation (3) vanishes and due to small velocities the viscous dissipation \( \mu \Phi \) may also be neglected. If we apply the Boussinesq approximation \( \rho(T) = \rho_0 [1 - \beta(T - T_0)] \) for the density variation in the buoyant term and use the linearized equation of state \( M(T,H) = \chi H_{0} - K(T - T_0) \) for the magnetization, the governing equations may be written

\[ \nabla \cdot \mathbf{u} = 0 \quad \cdots (4) \]
\[ \rho_0 \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p - \rho_0 \beta(T - T_0) \mathbf{g} \cdots (5) \]
\[ + \mu \nabla^2 \mathbf{u} - \mu_0 K(T - T_0) \nabla H \]
\[ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{k}{\rho c_{v,H}} \nabla^2 T \quad \cdots (6) \]

Magnetic field is produced by letting a constant current \( I \) through the inner cylinder. The magnitude of magnetic field and magnetic field gradient may then be written

\[ H_0 = \frac{I}{2\pi r} \quad \nabla H = -\frac{I}{2\pi r^2} e_r \quad \cdots (7) \]
Numerical Simulations

The simulations were carried out for the 0.1 m high cylinder with the diameters of the inner and the outer cylinder equal to 0.010 and 0.022 m, respectively.

Prandtl number is a dimensionless number, which may be used to estimate the ratio of thermal and hydrodynamic boundary layer thickness. The free convection hydrodynamic boundary layer is made up of two parts. One in which the velocity rises and a region where the velocity decays to zero. In the case of magnetic fluids, when the Prandtl number is much more than unity, the thermal boundary layer is thinner than the hydrodynamic one. Still it is important to construct the mesh in a way that both parts of hydrodynamic boundary layer as well as thermal boundary layer are captured. In Fig. 2 the $z$-velocity distribution and the temperature distribution with gravitational and magnetic Rayleigh numbers equal to $7 \times 10^5$ and 3000, respectively, are shown.

In the simulations 40 grid points were used in radial direction. In the light of velocity and temperature boundary layers shown in Fig. 2, the amount of grid points in radial direction seems to be sufficient for the Rayleigh numbers under consideration. The number of grid points was 200 in the azimuthal and 150 in the vertical direction, leading to hexahedral mesh with 1 200 000 cells.

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The physical properties of the studied ferrofluid were as follows: density $\rho = 1400$ kg/m$^3$, thermal expansion coefficient $\beta = 0.0002$ 1/K, kinematic viscosity $\nu = 1.3 \times 10^{-6}$ m$^2$/s, thermal diffusivity $\alpha = 1.3 \times 10^{-6}$ m$^2$/s, pyromagnetic coefficient $K = 200$ A/mK and the Prandtl number $Pr = 10$. All physical properties were considered constant and determined at average temperature and magnetic field.

Constant temperature boundary conditions were used for cylindrical surfaces. Top and bottom of the cylinder were insulated and no-slip boundary conditions were applied on all of the walls. Current $I$ led through the inner cylinder and temperature difference between the cylinders was varied in order to achieve the desired gravitational and magnetic Rayleigh numbers.

As an initial condition for the time dependent calculations the base state in the absence of fluid motion $\mathbf{u} = 0$ was used. In the considered cylindrical geometry the solving of conduction equation leads to the radial temperature distribution of the form:

$$T(r) = T_h - \frac{T_h - T_c}{\ln \left( \frac{D}{d} \right)} \ln \left( \frac{2r}{d} \right)$$  \hspace{1cm} (8)

Characteristic time for reaching the quasi-steady state may be approximated$^2$ as $t_0 = L^2/\nu$. In the analysis, the gravitational Rayleigh number $Ra$ and the magnetic Rayleigh number $Ra_m$ were defined as follows:
where $g$ is the acceleration of gravity, $\mu_0$ is the vacuum permeability, $L$ is a characteristic length equal to the gap width $L = (D - d)/2$ and $\Delta T$ and $\Delta H$ are the temperature and magnetic field differences over the gap, respectively. In the simulations, both gravitational and magnetic Rayleigh numbers were varied between 0 and $10^6$.

Results and Discussion

Simulations in the absence of gravity

At first, two-dimensional simulations were carried out for pure magnetic convection. In the absence of gravity two-dimensional simulations could reproduce the expected phenomena with counter rotating convection cells with diameter equal to the thickness of fluid layer. However, the onset of convection was found already with magnetic Rayleigh numbers about one thousand, which is much less than the theoretical $Ra_{m,cr} \approx 1800$. Three-dimensional simulations were carried out in order to increase the accuracy of the two-dimensional simulations. In Fig. 3, the average Nusselt number for the cylinder as a function of magnetic Rayleigh number has been presented. Dashed line in Fig. 3 represents the experimental correlation for average Nusselt number $Nu_{av} = 0.25Ra_m^{0.24}$.

It may be said that the results of three-dimensional steady-state simulations are in better agreement with the theoretical and experimental predictions than two-dimensional, but the simulations are very time consuming near the onset of instability, which makes the accurate prediction of critical Rayleigh number difficult.

Simulations in the presence of gravity

When the gravity perpendicular to the magnetic field gradient is introduced, the case becomes more complicated. The flow will be three-dimensional and in addition the flow may become unsteady consisting of upward drifting cells. Similar phenomena can be found when the buoyancy driven convection of ordinary fluids is studied in a vertical annulus. The cellular structure, which is stationary in a Cartesian geometry, drifts upward when the curvature is introduced. In natural convection the speed of rising cells has been found to depend on the radius and aspect ratios of the cylinders as well as on the fluid properties, namely Prandtl number. In this study only the effect of the magnetic and gravitational Rayleigh numbers and their ratio, $N = Ra_m/Ra$, on the convection was studied and other parameters, such as Prandtl number, radius ratio and aspect ratio, were kept constant.

The limiting values for the onset of pure buoyancy driven convection in a vertical annulus are $Ra > 5 \times 10^3$ and $Ra_m = 0$, and for pure thermomagnetic convection in the absence of gravity, $Ra_m > 1880$ and $Ra = 0$. Based on the simulations, these values may be considered also as critical values for combined convection. There was no clear evidence that any combination of gravitational and magnetic Rayleigh numbers, both being smaller than these critical values, would give a significant increase to the heat transfer rate of the pure conduction. More like it looked that for small gravitational Rayleigh numbers the introduction of magnetic field gradient may stabilize the flow. Though, once again it must be said that the simulations near the onset of instability are very time consuming, and the number of trial cases was too small to accurately determine the critical values for the onset of combined convection.

Unsteady simulations revealed that the speed of rising cells decreases when the ratio of magnetic and gravitational Rayleigh numbers is increased and for...
the ratio $N > 100$ the steady azimuthal convection, similar to that of pure thermo-magnetic convection, takes place. In the other end, when $N < 0.05$, the convection may be considered to be buoyancy dominated and the effect of magnetic field may be neglected. Temporal evolution of the flow in a vertical plane of the annulus for the Rayleigh numbers $Ra = Ra_m = 10^4$ is shown in Fig. 4.

However, the unsteadiness of the flow doesn't have much of an influence, when the average Nusselt numbers are calculated. In Fig. 5 the average Nusselt numbers as a function of gravitational and magnetic Rayleigh numbers have been plotted for selected cases. The values have been calculated based on a single time step after a quasi-steady state has been reached. Also some reference values for buoyancy driven convection, are shown.

Computer simulations of the three-dimensional thermo-magnetic convection have been performed in order to have a better understanding from the onset of magnetic convection and the relationship between the gravitational and magnetic convection. Necessary terms to describe the magnetic body force have been applied to the equation of momentum.

In the absence of gravity, the flow consists of counter rotating convection cells in a horizontal plane. Two-dimensional computer simulations revealed the expected flow pattern and the magnitude of average heat transfer rate was close to the expected values. However, the two-dimensional simulations couldn't predict the correct value for the critical Rayleigh number with which the onset of convection occurs. Conducting the three-dimensional simulations improved the results somewhat, although the accurate simulation of the exact moment of the onset of instability is very time consuming. Recently Zebib studied the critical stages and stability of the thermal convection of magnetic fluid in a three dimensional cylindrical annulus. His analysis agreed well with the microgravity experiments of Odenbach. In the analysis it was shown that there are competing states for the onset of instability. The linear theory predicts the onset of convection to be three-dimensional, with $Ra_m = 1802.36$ and the two-dimensional azimuthal convection takes place when the magnetic Rayleigh number is further increased beyond $Ra_m > 1900$.

In the presence of gravity, the convective motion will be three-dimensional and in addition it may be time dependent with cellular patterns drifting upwards. Reason for this phenomenon is thought to be the curvature of the cylindrical surface, which destroys the symmetry of the velocity profile. The simulations showed that the introduction of magnetic field gradient may stabilize the flow with small gravitational Rayleigh numbers and that convection may occur only, if either the critical magnetic Rayleigh number of pure thermo-magnetic convection or the critical Rayleigh number of pure buoyancy driven convection is exceeded. The speed of rising cells decreases when the ratio of magnetic and gravitational Rayleigh numbers increases, leading to the steady azimuthal convection, when $N > 100$.

**Conclusions**

Linear stability analysis of the buoyancy driven convection in a cylindrical annulus has revealed that
the magnitude of Prandtl number has a major effect on the stability of the flow. As for low Prandtl numbers increasing the curvature of the annulus stabilizes the flow whereas the opposite is true for high Prandtl numbers\textsuperscript{1}. Prandtl numbers of a magnetic fluid vary greatly and magnetic fluids with very high Prandtl numbers are often considered. Therefore, it is important to extend the future studies to include the effect of Prandtl number and radius ratio of the cylinders. There exist a number of possible applications for the thermo-magnetic convection of magnetic fluids. Two fields, where the advantages are obvious, are the micro-gravity applications, where the buoyancy driven convection is absent and the cooling of electronic components, where the temperature and magnetic field gradients like those presented in this study, are inherent.

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