Loss factor for a clamped edge circular plate subjected to an eccentric loading

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Present article deals with the analytical evaluation of modal loss factor of a clamped edge thin circular plate of uniform thickness subjected to an eccentric harmonic point load. Starting with the fundamental equation of forced transverse vibration of a thin elastic plate in terms of Green’s function; the expression for the deflections is determined. Stresses are calculated with the help of general stress-deflection equations and the modal loss factor is evaluated at resonance condition. The nodal circles’ radii are also determined analytically for each mode shape.

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The material damping\textsuperscript{1}, which is also known as hysteretic damping or structural damping, plays an important role in the control of the vibratory response of structural members. It is also of importance in the acoustical field to control the sound radiation from members like plates. It is measured in terms of modal loss factor. Lazan\textsuperscript{2} has discussed the procedure for the evaluation of the internal loss factor of various members.

Leissa\textsuperscript{3} has reviewed the vibration response of plates. Morse and Ingard\textsuperscript{4} have also studied the vibration response of a clamped, thin circular plate of uniform thickness with eccentric point load. Nigam, et al.\textsuperscript{5} have evaluated the modal loss factor for a rectangular plate of variable thickness. Little work is available in literature on analytical evaluation of material damping of thin circular plates of uniform thickness with an eccentric harmonic point load.

The schematic representation of a clamped edge, thin, elastic, homogeneous, and isotropic circular plate of uniform thickness subjected to an eccentric harmonic point load is shown in Fig. 1. A polar coordinate axis system is taken for the analysis. The classical small deflection thin plate theory has been employed. To account for energy dissipation, rigidity has been taken in complex form\textsuperscript{6}. Starting with the fundamental equation of forced transverse vibration of thin elastic plate in terms of Green’s function\textsuperscript{7}; the expression for the deflections is determined. Once the deflection is known, the stresses are calculated with the help of general stress-deflection equations and the modal loss factor\textsuperscript{2} is evaluated at resonance condition. Also, the nodal circles’ radii are determined for each mode shape.

**Forced Vibration of Plate**

Consider a thin circular plate of radius ‘\(a\)’ clamped at edge subjected to a harmonic point load ‘\(P\)’ as shown in Fig. 1. The damping is taken into account by considering flexural rigidity to be of complex form. For a thin circular plate of isotropic and homogeneous material clamped at edge subjected to a harmonic unit point load, the classical equation of motion given by Morse and Ingard\textsuperscript{4} in terms of Green’s function is:

\[
\begin{align*}
B\nabla^4 G + 2\rho h \frac{\partial^2 \phi}{\partial t^2} G &= e^{-j\omega t} \delta(r-r_0) \delta(\phi-\phi_0) \\
\end{align*}
\]

where

\[
G = G(r, \phi | r_0, \phi_0), \quad [\text{the deflection at } (r, \phi) \text{ due to unit load at } (r_0, \phi_0)]; \quad B = \frac{2Eh^3}{3(1-\nu^2)} \quad \text{(Flexural rigidity)}; \quad E = \text{Modulus of elasticity}; \quad \rho = \text{Mass density of the material of the plate}; \quad \nu = \text{Poisson ratio}; \quad h = \text{Half of the thickness of the plate}; \quad t = \text{Time}; \quad \text{and, } \omega = \text{Frequency of excitation.}; \quad \nabla^4_r = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right).
\]

Let the harmonic point load acting on the plate be given by:

\[
P(r_0, \phi_0, t) = P(r_0, \phi_0) e^{-j\omega t}
\]

\[
\text{... (2)}
\]
Let the harmonic response of the plate due to the harmonic excitation be given by:

$$G(r, \phi) = G(r, \phi) e^{-j\omega t} \quad \ldots (3)$$

For a steady state condition, from relations (1), and (3), one gets:

$$\nabla^4 G - \gamma^4 G = \frac{1}{r} \varphi \delta(r - r_0) \varphi \delta(\phi - \phi_0) \quad \ldots (4)$$

where,

$$\gamma^4 = \frac{3 \rho \omega^2 (1 - \nu^2)}{Eh^2} \quad \ldots (5)$$

Let us consider the homogeneous equation for a circular plate:

$$\nabla^4 G - \gamma^4 G = 0 \quad \ldots (6)$$

The solution of homogeneous Eq. (6) in terms of Bessel's functions is given by:

$$G = A J_m(\gamma mn r) + B I_m(\gamma mn r) \quad \ldots (7)$$

where, $A^*$ and $B^*$ are constants, $\gamma mn$ is a frequency parameter, suffices ‘m’ and ‘n’ in frequency parameter stand for nodal diameter and nodal circle, respectively, $J_m$ is the m-th order Bessels’ function of first kind and $I_m$ is the m-th order modified Bessels’ function of first kind.

For a clamped edge circular plate of radius ‘a’, boundary conditions are given by:

$$G = 0 \quad \text{at } r = a$$

$$\frac{\partial G}{\partial r} = 0 \quad \text{at } r = a \quad \ldots (8)$$

From the first boundary condition, one gets:

$$B^* = -A^* \frac{J_m(\gamma mn a)}{I_m(\gamma mn a)} \quad \ldots (9)$$

and, the second boundary condition gives:

$$\frac{d}{dr} \left[ J_m(\gamma mn r) \right]_r=a = 0$$

On solving Eq. (10), one gets:

$$\gamma mn = \frac{\pi}{a} \beta mn \quad \ldots (11)$$

where,

$$\beta mn \quad \ldots \quad \ldots (12)$$

Also, for $n \to \infty$

$$\beta mn = \frac{m}{2} + n$$

The characteristic function is given by:

$$\Psi_{c mn}(r, \phi) = \left\{ \begin{array}{ll} \cos m\phi & \text{if } \Psi_{m} \cos \sigma \text{ is cos term} \\ \sin m\phi & \text{if } \Psi_{m} \cos \sigma \text{ is sin term} \end{array} \right\}$$

$$\frac{d}{dr} \left[ J_m(\gamma mn r) \right]_r = 0$$

From Eqs (4) and (13), one gets:

$$\sum_{skl} A_{skl} \left( \gamma^4_{kl} - \gamma^4 \right) \Psi_{skl}(r, \phi) = -\frac{1}{r} \delta(r - r_0) \delta(\phi - \phi_0) \quad \ldots (14)$$
Considering the property of the Delta function and multiplying Eq. (14) by \( \Psi_{mn}(r, \phi) \) \( r \, dr \, d\phi \) and integrating it over the plate area, one gets:

\[
G = \frac{1}{\pi a^2} \sum_{mn} \frac{\Psi_{mn}(r, \phi) \Psi_{mn}(r_0, \phi_0)}{\Lambda_{mn} \left( r_0^2 - r^2 \right)} \quad \ldots (15)
\]

where,

\[
\left[ \int_0^{2\pi} \int_0^a \left( \Psi_{mn}(r, \phi) \right)^2 \, r \, dr \right] = \pi a^2 \Lambda_{mn}, \quad \ldots (16)
\]

and,

\[
\Lambda_{mn} = \frac{2}{\varepsilon_m} \left[ (J_m(\gamma_{mn} a))^2 + (J_m^*(\gamma_{mn} a))^2 \right] \quad \ldots (17)
\]

For a particular mode \((m, n)\), the stress in the plate will be maximum when the deflection due to the harmonic point load is maximum at the resonance i.e. when \( \gamma = \gamma_{mn} \). At resonance, Eq. (15) reduces to:

\[
G_{mn} = \frac{1}{\pi a^2} \frac{\Psi_{mn}(r, \phi) \Psi_{mn}(r_0, \phi_0)}{J \eta \gamma_{mn}^4 \Lambda_{mn}} \quad \ldots (18)
\]

where, \( \eta \) = Modal loss factor; \( B^{**} = B(1 + j \eta) \), Flexural rigidity in complex form; \( G_{mn} = G \), For \((m, n)\) mode; and, \( \frac{1}{j} \) Phase difference between the excitation and the response and it is neglected in further analysis.

Thus,

\[
G_{mn} = \frac{1}{\pi a^2} \frac{\Psi_{mn}(r) \Psi_{mn}(r_0) \cos m(\phi - \phi_0)}{\eta \gamma_{mn}^4 \Lambda_{mn}} \quad \ldots (19)
\]

where,

\[
\Psi_{mn}(r) = J_m(\gamma_{mn} r) - J_m(\gamma_{mn} a) \frac{I_m(\gamma_{mn} a) I_m(\gamma_{mn} r)}{I_m(\gamma_{mn} a)} \quad \ldots (20)
\]

\[
\Psi_{mn}(r_0) = \Psi_{mn}(r) \quad \text{at} \quad r = r_0 \quad \ldots (21)
\]

The deflection at any point \((r, \phi)\) in the plate due to eccentric harmonic point load \( P e^{-j\omega t} \) is given by:

\[
w(r, \phi) = \frac{P}{B} G_{mn} \quad \ldots (22)
\]

Substituting the value of \( G_{mn} \) from relation (19) in Eq. (22), one gets:

\[
w(r, \phi) = \frac{P}{\pi a^2 \eta \gamma_{mn}^4 \Lambda_{mn}} \frac{\Psi_{mn}(r) \Psi_{mn}(r_0) \cos m(\phi - \phi_0)}{J \eta \gamma_{mn}^4 \Lambda_{mn}} \quad \ldots (23)
\]

Further simplifying Eq. (23) by substituting the values from Eqs (22) and (23) leads to:

\[
w(r, \phi) = \frac{C_{mn}}{\eta} \{ J_m(\gamma_{mn} r) - R_{mn} I_m(\gamma_{mn} r) \} \cos m(\phi - \phi_0) \quad \ldots (24)
\]

where,

\[
C_{mn} = \frac{P}{\pi a^2 B \gamma_{mn}^4 \Lambda_{mn}} \frac{\Psi_{mn}(r_0)}{J \eta \gamma_{mn}^4 \Lambda_{mn}}
\]

\[
R_{mn} = \frac{J_m(\gamma_{mn} a)}{I_m(\gamma_{mn} a)}
\]

Also, modal frequency from Eqs (5) and (11) will be:

\[
\nu_{mn} = \frac{\pi h}{2a^2} (\beta_{mn})^2 \sqrt{\frac{E}{3\rho(1-\nu^2)}} \quad \ldots (25)
\]
Loss Factor for a Plate

The plate is conceptually thought to consist of a large number of small elements of area \( dr \, d\phi \). Stresses \( \sigma_r \), \( \sigma_\phi \), and \( \tau_{r\phi} \) at each element are obtained from simple stress displacement relations. From these stress values, principal stresses \( \sigma_{a_1} \) and \( \sigma_{a_2} \) at the element are evaluated. There are two criteria for determining the loss factor values. One is based on dilatational strain energy and other is based on distortional strain energy. The former gives the upper bound values and the latter gives the lower bound values for the modal loss factor. An equivalent stress \( \sigma_e \) can be obtained at the center of each element for the two criteria. The loss factor can then be obtained from:

\[
\eta = \frac{J E}{\pi} \sum \frac{\sigma_e^N}{\sigma_e^2} \quad \quad \text{(26)}
\]

where, \( \sum \) is carried through elements and \( J \) and \( N \) being the damping constants of the material. The equivalent stress \( \sigma_e \), based on the dilatational strain energy criterion can be obtained as:

\[
\sigma_e = \sigma_{a_1} \left(1 - 2\nu \xi + \xi^2\right)^{1/2} \left(1 + \xi\right)^{N/2} \quad \quad \text{(27)}
\]

Based on distortional strain energy criterion, the equivalent stress is given as:

\[
\sigma_e = \sigma_{a_1} \left(1 - 2\nu \xi + \xi^2\right)^{1/2} \left(1 - \xi + \xi^2\right)^{N/2} \quad \quad \text{(28)}
\]

Results and Discussion

As an illustration, a plate of SAE1020 steel having radius, thickness, mass density, modulus of elasticity, and Poisson ratio as 0.6 m, 0.003 m, \( 7.6 \times 10^3 \) kg/m\(^3\), \( 2.068 \times 10^{11} \) N/m\(^2\), and 0.3, respectively, subjected to a harmonic point load of 150 N has been considered. The damping properties \( J \) and \( N \) for the above plate material are \( 3.04 \times 10^{-18} \) and \( 2.286 \), respectively. Modal loss factor for different modes have been computed for the various eccentricity, i.e. \( r_0 / a \) and discussed to support the present method of analytical evaluation of modal loss factor. Comparing the radii of the nodal circle(s) with the values given by Leissa also validates the work presented in this paper.

The modal loss factor depends on the mode shape. From stress-displacement relationship, stress at a point is maximum when the deflection is maximum and vice-versa. Hence, for a particular mode, the modal loss factor will be maximum when the load acts through an antinode i.e. the point of maximum deflection and will be minimum when it acts through a node. Tables 1 and 2 show the variation of modal loss factor with the eccentricity for different modes. Table 1 is based on the distortional strain energy criterion, which gives a lower limit of the modal loss factor. Table 2 is based on the dilatational strain energy criterion, which gives an upper limit of the modal loss factor.

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Table 2—Variation of modal loss factor with eccentricity based on dilatational strain energy criterion

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Table 3 — Comparison of radii of nodal circle(s) for different mode shapes

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<td>Given by Leissa(^3)</td>
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Fig. 1 — Circular plate with an eccentric harmonic point load

Fig. 2 — Variation of modal loss factor with eccentricity based on distortional strain energy criterion

Fig. 2 shows the effect of variation of eccentricity on the modal loss factor. It is observed that the modal loss factors for higher modes are lower than the fundamental mode. The minimum value of the modal loss factor for a mode gives location of the nodal circle(s) for that particular mode. A comparison of radii of the nodal circle(s) with the values given by Leissa\(^3\) is given in Table 3.
Further, the modal loss factor also depends upon the thickness of the plate. The effect of variation of thickness of the plate on the modal loss factor is shown in Fig. 3. The modal loss factor decreases hyperbolically with increase in the thickness of the plate.

Conclusions
The present work for analytical evaluation can be used to obtain an estimate of modal loss factor for a clamped edge thin circular plate subjected to an eccentric harmonic point load. The dilatational strain energy criterion and the distortional strain energy criterion give the upper and lower values of the modal loss factor, respectively for a clamped edge thin circular plate subjected to an eccentric harmonic point load. The actual value of the modal loss factor for practical purpose would be in between the two limits. Further, the minimum value of the modal loss factor for a mode gives location of the nodal circle(s) for that particular mode.

References
2 Lazan B J, Damping of materials and members in structural mechanics (Pergamon, New York), 1968.