Effect of errors in position coordinates of the receiving antenna on single satellite GPS timing

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It is well established that the position co-ordinates of the receiving antenna should be determined precisely in advance for getting time of the GPS receiver through a single GPS satellite technique. So it is desirable to know the extent of accuracy of the position coordinates required for a particular timing accuracy. In this paper, an analytical expression relating to the position error and the corresponding error in the time of the GPS receiver has been derived. Time of the GPS receiver error caused by the error in position coordinates largely depends on the position of the satellite indicated by the respective elevation and azimuth of the satellite. To validate the derived formulation, it is important to configure the experimental plan judiciously. A special experiment has accordingly been conducted at National Physical Laboratory, New Delhi, India (NPLI). The observed data have been found to tally well with the derived relation.

Keywords: Satellite, GPS timing, Antenna, Single satellite GPS timing

1 Introduction
GPS has evolved into a very powerful time transfer technique with two important features. Time of the GPS receiver provides very high accuracy of few nanoseconds and Time of the GPS receiver has wide coverage as it is accessible anywhere in the world and anytime of the day. The time scale of US Naval Observatory (USNO) is technically named as Universal Coordinated time of USNO in short UTC(USNO). All clocks of GPS satellites are synchronized to UTC(USNO). Basic principle\(^1\) of GPS is based on the pseudo ranges of the satellite which are measured at the receiving end by the travel time of the GPS signal. GPS signal carries much information including the orbital parameters of the corresponding satellites. From these parameters, the instantaneous position co-ordinates of the satellite are predicted. Basic purpose of GPS service is to provide the instantaneous values of three position co-ordinates of the user and the time offset of the receiver clock. Determination of these four unknown parameters demands simultaneous measurements from four different GPS satellites. So for normal applications (i.e. both for position and time), one needs to receive signals from minimum four GPS satellites simultaneously. Mostly common GPS receivers work on this principle.

There exists another technique\(^2-4\) of using GPS for the timing application. If three coordinates of the receiver position are known in advance then one may get time just by the measurement from a single satellite. This paper concentrates on this mode of GPS operation. This mode is mainly used by time keeping laboratories and the antenna has to be placed at a fixed known point. The special timing receiver with the single satellite technique was developed right from the inception of the GPS. In course of time, the single satellite timing receiver has gone through many phases of development both in terms of hardware technology and software improvisations. This special type of timing receiver is currently used widely by timing keeping laboratories contributing to the generation of UTC coordinated by BIPM.

For getting Time of the GPS receiver through a single GPS satellite, it is desirable to know the extent of accuracy of the position coordinate required for a particular timing accuracy. But no study has been reported to ascertain quantitatively the effect of position error on the timing accuracy by this technique and its dependence on the relative position of the satellite. This paper attempts to find an analytical expression on the effect of the position error on Time of the GPS receiver relating the corresponding satellite position. A special experiment has also been conducted at National Physical Laboratory, New Delhi, India (NPLI) to corroborate the derived relation. The observed data have been analyzed and presented in this paper.
2 Analytical Approach

The error in GPS timing operated in the single satellite mode is directly correlated to the range error arising out of the error in position coordinates of the receiving antenna. In order to derive an analytical expression of the correlation, let us assume that in earth centered fixed coordinate systems of WGS84, the exact coordinates of the receiving antenna are \( x^0, y^0 \) and \( z^0 \). The coordinates have been determined in advance and are fed to the GPS receiver in advance (say as \( x, y \) and \( z \)). Thus, \((x-x^0, y-y^0 \) and \( z-z^0 \)) represent the error in the position coordinates.

Because of the error in the coordinates, there will be range error leading to the timing error. Now let us assume that \( S \) is the position of GPS satellite in horizontal coordinate system with respect to the actual position \( O \) of the receiving antenna as the origin (Fig.1).

\( X \) and \( Y \) axes lie north-ward and east-ward, respectively and \( Z \) axes towards zenith. Let \( P \) is the position of the antenna as determined. \( \overrightarrow{OP} \) becomes the Position error vector \( \overrightarrow{E} \). \( OP \) is very small compared to \( OS \). Therefore, it is quite logical to assume that \( OS \) and \( PS \) are parallel. \( \overrightarrow{PO}' \) is projection of \( \overrightarrow{OP} \) on satellite range vector \( \overrightarrow{OS} \). Thus, \( \overrightarrow{OP}' \) is range error \( \Delta R \) (i.e. \( \Delta R = \overrightarrow{OP}' = \overrightarrow{OP} \cos \Phi \)) caused by the position error.

\[
\Delta R = |\overrightarrow{E}| \cos \Phi = |\overrightarrow{E}| \frac{\overrightarrow{S} \cdot \overrightarrow{E}}{R} = uS \cdot \overrightarrow{E} \quad \ldots(1)
\]

\( uS \) is the Unit vector of the direction of satellite \( S \) from the origin \( O \).

Satellite \( S \), has Elevation \( (e) \) and Azimuth \( (a) \) and Range \( (R) \) with respect to the receiver position. In order to evaluate the \( \Delta R \), it is necessary to transform the vectors to horizontal coordinate system as shown in Fig. 1. The satellite position represented by \( e, a \) and \( R \) may be transformed to the horizontal system as:

\[
\overrightarrow{\tilde{S}} = \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix} = R \begin{bmatrix} \cos(e) \cos(a) \\ \cos(e) \sin(a) \\ \sin(e) \end{bmatrix} = R \overrightarrow{S} = Ra \overrightarrow{S} \quad \ldots(2)
\]

Similarly, the errors in position coordinates \((x-x^0, y-y^0 \) and \( z-z^0)\) may also transformed as :

\[
\overrightarrow{E} = \begin{bmatrix} -\cos(L) \sin(B) \\ -\sin(L) \cos(B) \end{bmatrix} - \begin{bmatrix} \sin(L) \cos(B) \\ \sin(L) \sin(B) \end{bmatrix} \begin{bmatrix} x-x^0 \\ y-y^0 \end{bmatrix} = \begin{bmatrix} E_x \\ E_y \end{bmatrix} \quad \ldots(3)
\]

where \( L \) and \( B \) are the latitude and the longitude of the exact position of the antenna \( (O \) in Fig.1).

By combining Eqs (1-3), the range error may be given by:

\[
\Delta R = E_x \cos(e) \cos(a) + E_y \cos(e) \sin(a) + E_z \sin(e) \quad \ldots(4)
\]

In single satellite GPS mode, the corresponding time error \( \Delta T \) in nanoseconds) will be directly related as:

\[
\Delta T = \frac{\Delta R}{c} = \frac{1}{30} [E_x \cos(e) \cos(a) + E_y \cos(e) \sin(a) + E_z \sin(e)] \quad \ldots(5)
\]

Here, \( \Delta R \) is in cm

From Eq. (3), it follows that:

\[
E_x = -\cos(L) \sin(B) \Delta x - \sin(L) \sin(B) \Delta y + \cos(B) \Delta z \quad \ldots(6)
\]

\[
E_y = -\sin(L) \Delta x + \cos(L) \Delta y \quad \ldots(7)
\]

\[
E_z = \cos(L) \cos(B) \Delta x + \sin(L) \cos(B) \Delta y + \sin(B) \Delta z \quad \ldots(8)
\]

where \( \Delta x = x-x^0, \Delta y = y-y^0, \Delta z = z-z^0\)

From Eq.(5) coupled with Eqs (6-8), the timing error \( \Delta T \) may be determined.
3 Experimental Details

To validate the above formulation, it is important to configure the experimental plan very carefully. If we measure the difference between the Time of the GPS receiver and the reference standard time, the difference becomes the error in Time of the GPS receiver (dT). It is important to note that this error is contributed not only by the error in coordinate of the receiving antenna (e_pos) but also by the error of the satellite ephemeris (\(e_{eph}\)), ionosphere (d_ion) and troposphere (d_trop) model errors, multipath error (\(\alpha_{mul}\)), receiver noise (r_nos) and the satellite clock error (d_tsat) as given by:

\[
c \cdot dT = e_{pos} + e_{eph} + c \cdot d_{trop} + d_{ion} + \alpha_{mul} + r_{nos}
\]

It is almost impossible to separately quantify the contribution to the time error by the respective factors. So the contribution of the position error to Time of the GPS receiver error cannot be found out from the measurement of one GPS receiver. To circumvent this, a special experiment has been planned. The corresponding experimental set-up is shown in Fig. 2. Two special GPS timing receivers (say, Receiver A and Receiver B) of the same make are used. Two antennas are placed side by side (separated by the distance of only 3.2 meters, in this case) and their coordinates are known precisely.

Two receivers used here are special in the sense that they have built in time interval counters with a resolution of 100ps. They compare the Time of the GPS receiver with respect to 1 pulse per second (1 pps) externally fed from the reference clock. In this set up, both receivers are fed with 1pps from the common source of the master Cesium clock (high performance) of NPLI. Each receiver records the time difference between Time of the GPS receiver and the externally fed 1pps (say for receiver A denoted as (Ref-GPS_A), the elevation (e), azimuth (a) and PRN number of the corresponding satellite and the time and date of the observation. The data from the two receivers are recorded in the computer for further analysis. Two receivers record data independently. For analysis, data from both the receivers are sorted for strict common view (i.e. for common satellite at the same time of observation). From the common view data, values of the (Ref-GPS_A) – (Ref-GPS_B) are calculated. They are denoted by \(\delta t\). As two antennas are almost collocated, the errors in ionosphere and troposphere delay would be exactly the same for Receiver A and Receiver B in strict common view. Two antennas are located at a place whose surroundings are quite clear eliminating the chance of multipath. So \(\delta t\) would be contributed only by the noise of the two receivers and by the error of the coordinates of the two antennas. So \(\delta t\) may be derived from Eq. (9) as:

\[
c \cdot \delta t = c \cdot (dT_A - dT_B) = e_{pos}(A) - e_{pos}(B) + r_{nos}(A) - r_{nos}(B)
\]

![Fig.2 — Experimental set-up for the special experiment campaign](image-url)
3.1 Analysis of the observed Data

Position Coordinates of two antennas have been determined very precisely with respect to IGS network through a special campaign\(^6\). This method requires a special dual frequency geodetic receiver. Measurements of pseudo range were recorded continuously for a sufficient amount of time. These records were analyzed in DGPS mode with respect to a nearby IGS station of NGRI, Hyderabad. The analyses made use of the IGS products like ionosphere and troposphere delays and satellite orbits. The position coordinates were determined within an accuracy of the order of few centimeters.

These coordinates are fed to the respective receivers. \(\delta t\), \(e\), \(a\) and the corresponding time of observations are recorded for few days. The values of the \(\delta t\) have been found to have 1\(\sigma\) of 2.5 ns which, according to Eq. (10), may be primarily contributed by the noise of the receiver assuming the position coordinates have no error.

One way to validate the formulation given in Eq. (5), is to introduce deliberate error in position coordinates in one of the antennas substantially so that the contributions to time error by the errors of the position coordinates is much higher than that (~2.5 ns) by the receiver noise. In order to satisfy this condition, two special observations were planned. For one set of observations, the receiver \(B\) is fed with a new set of coordinates which have deliberately moderate errors. The second set was with another set of coordinates with large errors. In such a situation, \(\delta t\) will be primarily contributed by the position error of antenna \(B\), the other factors in Eq. (10) being comparatively insignificantly small. Thus, in such a situation, Eq. (10) reduces to:

\[
\delta t = \frac{e_{\text{pos}}(B)}{c} \quad \cdots (11)
\]

Case I — For one set of observation, position errors in receiver \(B\), introduced deliberately have been \(\Delta x = 49.89\) meters, \(\Delta y = -169.94\) meters \(\Delta z = 183.71\) meters. Feeding these values in Eqs (6), (7) & (8), \(\delta t\) (in nanoseconds) may be theoretically from Eq. (5) as:

\[
\delta t = \frac{1}{30}[-465134.599 \cdot \cos(e) \cdot \cos(a) \\
-472977.4533 \cdot \cos(e) \cdot \sin(a) \\
-3.61795810E + 03 \cdot \sin(e)] \quad \cdots (12)
\]

The variations of \(\delta t\) [based on Eq. (12)] with the azimuth for fixed values of the elevation are shown in Fig. 3. It is impossible to generate similar graphs with the experimental data, as it is quite unlikely to get a good amount of data corresponding to any particular value of elevation angle. Rather, it is more pragmatic to look for a large number of data for elevation angle within a certain range of values. Keeping this in mind, the experimental data, for example, has been sorted for all values of elevation in the range 50\(^\circ\)-70\(^\circ\) to validate the graphs of Fig. 3. The values of \(\delta t\), thus sorted, are plotted against the corresponding values of azimuth as shown in Fig. 4. The solid line in Fig. 4 is the corresponding theoretical values following Eq. (12) for the elevation angle of 60\(^\circ\) which is the mid value between 50\(^\circ\) and 70\(^\circ\). It is interesting to note that the theoretical line lies well inside the experimental data. Experimental data are for all
values of elevation between 50° and 70°. It is observed that there is a wider gap between the theoretical graphs for 50° and 70°, respectively of elevation particularly around the azimuth values of 0° and 360° as shown in Fig. 3. Because of this reason, wider scatter of experimental data near 0° of azimuth and also near the 360° of azimuth is shown in Fig. 4. Similar sorting of data have been made for corresponding data for elevation in the range 10°-30° and shown in Fig. 5. Interestingly, Fig. 5 shows the perfect match between the experimental data for elevation angle in the range 10°-30° and the theoretical line for elevation angle of 20°. This may be justified by the fact that theoretical lines for elevation angles of 10°, 20° and 30° of Fig. 3 are very close to each other.

Case II — For another set of observation, position errors in receiver B, are:

\[ \Delta x = 5099.47 \text{ meters}, \Delta y = 1091.78 \text{ meters}, \Delta z = 4100.07 \text{ meters} \]

For these values Eq. (9) reduces to:

\[
\delta t = \frac{1}{30} [23534.09335 \cos(\varepsilon) \cos(\alpha) - 8637.679238 \cos(\varepsilon) \sin(\alpha) - 4.765546786E + 03 \cdot \sin(\varepsilon)]
\]

Eq. (13) has been graphically shown in Fig. 6. In an attempt to match the experimental data with graphs as shown in Fig. 7, observations similar to case I were found as shown in Figs 7 and 8.

Critical inspections of the experimental data as discussed above exhibit excellent match with theoretical prediction. This leads one to confirm the
validity of the relation as shown in Eq. (5). Guided by Eq. (5), it is also easy to conclude that to achieve an uncertainty of \( \pm 2 \text{ ns} \), one must have the position coordinates known correctly within an error limit of 30 cm (i.e. \( \Delta x = \Delta y = \Delta z = 30 \text{ cm} \)).

4 Conclusions

For time transfer through single satellite GPS technique, the precise knoweldge of position coordinate is important. The relation between the position error and the corresponding calculated error in the Time of the GPS receiver has been derived. Time of the GPS receiver caused by the error in position coordinates largely depends on the position of the satellite indicated by the respective elevation and azimuth of the satellite. Observed data obtained through a specially planned experiment has been found to match the derived relation excellently.

It is well known that more accuracy is the requirement of the position coordinate more complicated the process of position determination becomes. So this formula gives a prior knowledge of position accuracy required for a planned time accuracy. Based on this, timing set-up may be configured.

References

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Annexure

Suggestions for further analysis

We have analyzed earlier keeping elevation constant. Let us try now keeping azimuth constant. Let us do as follows. I am attempting to find \( E_x, E_y \), and \( E_z \) from the experimental data.

\[
\Delta T = \frac{ \Delta R }{ c } = \frac{ 1 }{ 30 } [ E_x \cos(e) \cos(a) + E_y \cos(e) \sin(a) + E_z \sin(e) ]
\]

Case I

If \( \alpha = 0^\circ \) then

\[
\Delta T = \frac{ \Delta R }{ c } = \frac{ 1 }{ 30 } [ E_x \cos(e) + E_z \sin(e) ]
\]

Case II

If \( \alpha = 90^\circ \) then

\[
\Delta T = \frac{ \Delta R }{ c } = \frac{ 1 }{ 30 } [ E_y \cos(e) + E_z \sin(e) ]
\]

Case III

If \( \alpha = 180^\circ \) then

\[
\Delta T = \frac{ \Delta R }{ c } = \frac{ 1 }{ 30 } [ -E_x \cos(e) + E_z \sin(e) ]
\]

Case I

Generate experimental data \( \Delta T \) by sorting all values of azimuth between 350° and 10°. Plot \( \Delta T \) with respect to elevation.

Case II

Generate experimental data \( \Delta T \) by sorting all values of azimuth between 80° and 100°. Plot \( \Delta T \) with respect to elevation.

Case III

Generate experimental data \( \Delta T \) by sorting all values of azimuth between 170° and 190°. Plot \( \Delta T \) with respect to elevation.

If number of data is not sufficient, the range of azimuth may be extended to 20° instead of 10° as suggested earlier.

From three sets of data it may be attempted to find the fit of the equation shown above for each case. Do you have better software tools to have fit of the equation (not linear fit) in the data? The coefficients of the fit may give the values of \( E_x, E_y \), and \( E_z \).

You may try the above with A-B data and also only A data.