The pendula in mathematical sciences

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The role played by the pendulum and its variants in the development of mathematical sciences is reviewed. In particular, the case of a pendulum viz. harmonic oscillator in physics is highlighted in the context of mathematical, conceptual, conventional and engineering disciplines besides the one in mathematical physics. It is pointed out that in all these applications the concept of pendulum is used as a vehicle of knowledge mainly on the basis of structural analogy. Finally, its role is an example of the consciousness-manifesting phenomenon, will be discussed.

Keywords: Pendulum, Mathematical sciences, Conscious pendulum

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1 Introduction

Since the time of Newton (or perhaps earlier!) the pendula in various forms have been the subject of great importance not only in the development of mathematical tools but also in providing a rich theoretical framework within which a variety of natural phenomena could be made understandable. To a great extent, the role of pendula in these developments can be compared with that of a wheel in the day-to-day life. Obviously, if the discovery of wheel would not have been there then imagine what would be the style of our life in terms of luxuries. In fact, wheel and its variants have played an important role in providing a comfortable life to the mankind. In the same way, without the knowledge of pendulum many phenomena in nature would have remained unexplained and several theories in physics could not achieve their present status.

The simplicity of a pendulum lies in the mathematically idealized conditions that are imposed at the time of its designing, viz., (i) the string is assumed to be massless, (ii) the suspended mass (termed as ‘bob’) is assumed to be a (geometric) point mass. On the top of these two conditions, the third one is that (iii) during the oscillations the restoring force on the bob at any instant is linearly proportional to its displacement from the mean (equilibrium) position. In reality, however, there are several departures\(^1\) from this idealized model of simple pendulum.

In philosophical terms, if one believes in the three-world concept\(^2\) of Karl Poppar and John Eccles, then such a simple pendulum is bound to describe only a part of objective reality in nature. (According to Popper and Eccles, the objective reality inherent in a natural phenomenon can be considered at the level of three distinct ‘domains’, so called ‘worlds’ - one world as it exists in nature, the second as it is perceived by our sense organs and the third as it is described by modelling the phenomenon). The simple pendulum and it variants offer the case of the third world alone and that too with regard to the objective reality in nature. Absolute (or ultimate) reality however remains asymptotic even after accounting for all justifiable corrections to the simple pendulum.

By and large the use of pendula in different branches of knowledge has been there on the basis of structural analogy\(^2,4\) not only in the ‘objects’ and the ‘rules of the game’ but also at different space, time and mental scales. These contents or ingredients (at least the first two) of an analogy can further have (Ref. (2), Chap. 5) sub-contents like physical, geometrical or mathematical and philosophical, whereby providing a basis for the fine-tuning of the merit of the analogies. Interestingly, the simple pendulum while rich in mathematical sub-content will, however, be shown in this paper as to provide an equally viable vehicle of knowledge via its

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applications and variants with regard to other contents of analogy. In other words, it will be argued that it is the only case of a pendulum which offers a tool of its own type that has helped a lot in understanding nature at all space/time scales, i.e., from micro, macro, mega and to giga scales. Also, the pendulum alias (harmonic) oscillator has been the basis of a variety of physical theories.

2 Departures from Simplicity vis-a-vis Variants of a Pendulum/Oscillator

The well-known differential equation which governs the motion of a simple harmonic oscillator is given by

\[ \ddot{x} + \omega_0^2 x = 0 \]  \hspace{1cm} (1)

where \( x(t) \) is the displacement from the mean position at any time \( t \), and \( \omega_0 = \sqrt{k/m} \) is its frequency with \( k \) as the force constant and \( m \) as the mass of the bob. On the other hand, for a pendulum of length \( l \) oscillating in the field of gravity (as a rigid body) the governing equation is

\[ I \ddot{\theta} + mg \ell \sin \theta = 0 \]  \hspace{1cm} (2)

which, for the small \( \theta \) case, reduces to a linear version (since \( \sin \theta \approx \theta \) and the moment of inertia \( I = m \ell^2 \) ), namely

\[ \ddot{\theta} + \omega_0^2 \theta = 0, \quad \left( \omega_0 = \sqrt{g/\ell} \right) \]  \hspace{1cm} (3)

where \( g \) is the acceleration due to gravity.

Departures from the simplicity of a pendulum can arise (i) out of physical reality and/or (ii) out of mathematical generalization. (i) Out of physical requirements one can consider the cases when the string is massive and/or the bob is an extended object. In another case, one can consider the variations of the length (\( \ell \)) of the string and/or the mass (\( m \)) of the bob with respect to time \( t \), i.e., the case of a time dependent (TD) harmonic oscillator. Physically, for example, one can consider of a mechanical device by which the length of the pendulum can be made to change during the motion of the bob. i.e., \( \ell = \ell(t) \) case, or one can consider the motion of a small bucket (in place of bob) filled with ether which vaporizes during the motion, i.e., \( m = m(t) \) case. In either case the governing equation, after using a suitable change of variable, can be expressed as

\[ \ddot{x} + \omega^2(t) x = 0 \]  \hspace{1cm} (4)

The departure from simplicity of the pendulum also arises if one accounts for the damping (may be due to air resistance) in either of the time independent (TID) (cf. Eq. (1)) or the TD (cf. Eq. (4)) cases, leading particularly Eq. (4) to the form:

\[ \ddot{x} + b(t) \dot{x} + \omega^2 x = 0. \] \hspace{1cm} (5)

Eq. (5), in fact, after defining the new variable \( y(t) \) through \( x(t) = y(t) \exp \left[ -\int b(t)/2 \right] \), can again be recast in the form:

\[ \ddot{y} + \Omega^2(t) y = 0 \] \hspace{1cm} (6)

where \( \Omega^2(t) = \omega^2(t) - \frac{1}{2} \int b(t) \right] - \frac{1}{4} b^2 \). Note that in mathematical terms the study of Eq. (6) is similar to that of Eq. (4).

(ii) One case of departure from the simplicity of pendulum out of mathematical generalization is already listed above in Eq. (5) via the damping term that makes Eq. (4) a complete second order linear differential equation with variable coefficients. Further, generalization of Eq. (5) could be its inhomogeneous version, i.e., in physical terms when the time-varying external force \( f(t) \) is present, Eq. (5) becomes

\[ \ddot{x} + b(t) \dot{x} + \omega^2(t) x = f(t) \]  \hspace{1cm} (7)

Another departure in this spirit is through the presence of non-linear terms. Eq. (2), for example, corresponds to this case where the non-linearity arises by the presence of \( \sin^2 \theta \)-term. One can as well consider the non-linearity in terms of power function. (In fact, when the \( \sin^2 \theta \)-term is expanded in the powers of \( \theta \) for small \( \theta \)-values, this also offers the example of non-linearity of a particular type). As a matter of fact, a further generalized version of the non-linear oscillator can be described by the non-linear differential equation (NLDE)\(^6\).
In this context several non-linear (anharmonic) oscillators such as cubic, quartic, sextic have been of considerable interest in different branches of physics. In fact, the anharmonicity of quartic nature in terms of the concepts of Higgs mechanism or of second-order phase transitions has revolutionized some of the theories of physics.

Other dimensions of departure from the simplicity of the pendulum worth mentioning here are the cases of (i) singular pendulum, in which inverse power terms are present in addition to the harmonic potential, and (ii) coupled pendula/oscillators. When the motion of a single pendulum is considered in two or higher dimensions or of many (three or more) pendula are considered in one or higher dimensions, then the concept of coupling arises. In the former case, however, the forces have to be non-central and non-separable to give rise the coupling.

3 Pendula in Mathematics and Mathematical Physics

3.1 Pendula on the phase plane

For the phase plane studies, one uses the equivalent system of first-order equations corresponding to Eq. (1), namely

\[ \dot{x} = y; \quad \dot{y} = -\alpha_0 x^2, \]

which, on integration immediately yield

\[(1/2)y^2 + (1/2)\alpha_0 x^2 = \text{const} \]

implying a closed solution curve (ellipse) or trajectory on the phase (xy) plane. This describes the stable motion of the pendulum. On the other hand, if we proceed with the non-linear Eq. (2) and identify the equivalent system of equations as:

\[ \dot{\theta} = \eta; \quad \dot{\eta} = -(g/\ell) \sin \theta, \]

then on integration one obtains

\[(1/2)\eta^2 - (g/\ell) \cos \theta = \text{const} \]

which describes both closed and open solution curves on the \(\theta-\eta\) plane. The critical points surrounded by these curves are at \(\theta = n\pi, \quad n = 0, 1, 2, 3, \ldots\). Trajectories are closed (centre) or open (saddle) depending on whether \(\theta\) is even or odd multiple of \(\pi\). These phase portraits describe two distinct modes of motion of the pendulum, namely closed curves correspond to stable motion (the motion about the mean position) and the open ones correspond to unstable motion (a case of inverted pendulum) (see, Ref. (7), p. 291).

Next, we consider some other non-linear departures of the system given in Eq. (9). For simplicity, we choose \(\omega_0 = 1\) and consider the system (Ref. (7), Chapter 7),

\[ \dot{x} = -y + x(x^2 + y^2); \quad \dot{y} = x + y(x^2 + y^2), \quad \ldots (11) \]

\[ \dot{x} = -y - x(x^2 + y^2); \quad \dot{y} = x - y(x^2 + y^2). \quad \ldots (12) \]

Clearly, both these systems while have the same linearized version akin to harmonic oscillator, however, represent qualitatively different phase portraits in terms of their polar representations \((x = r \cos \theta, y = r \sin \theta)\), viz.,

\[ \dot{r} = r^3, \quad \dot{\theta} = 1 \quad \ldots (11A) \]

\[ \dot{r} = -r^3, \quad \dot{\theta} = 1 \quad \ldots (12A) \]

Eqs. (11A) and (12A) respectively represent unstable and stable spirals on the phase plane.

In another case, we consider the following non-linear departures from the simple pendulum (cf. Eq. (9)):

\[ \dot{x} = y + x(1 - r^2)/r; \quad \dot{y} = -x + y(1 - r^2)/r, \quad \ldots (13) \]

\[ \dot{x} = -y + x(2 - r - r^2)/r; \quad \dot{y} = x + y(2 - r - r^2)/r. \quad \ldots (14) \]

Note that the system given in Eq. (13) while admits a single limit cycle (an isolated closed curve) the system given in Eq. (14) admits two limit cycles on the phase plane.

As an example of the system given in Eq. (8) for \(\omega^2(t) = 1\), one can also consider the Van der Pol equation

\[ \dot{x} + \varepsilon (x^2 - 1) \dot{x} + x = 0 \]

as the departure from the harmonic oscillator with a non-linear damping term.

3.2 Coupled oscillators

We emphasize here only on two interesting cases of coupled oscillators which have been responsible to some major breakthrough in the development of mathematical physics.
(a) Henon-Heiles System: This system, studied first by Henon and Heiles is described by the Hamiltonian
\[ H = \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} (x_1^2 + x_2^2) + x_1^2 x_2 - \frac{1}{3} x_2^3 \] …(16)
and deals with four first-order differential equations as the equivalent system, viz., \( y_1 = p_1, \ y_2 = p_2 \)
\[ \dot{x}_1 = y_1; \dot{y}_1 = -x_1 - 2x_1x_2; \dot{x}_2 = y_2; \dot{y}_2 = -x_2 - x_1^2 + x_2^3. \] …(17)

The system given in Eq. (17) as such is found to describe the chaotic motion. However, even a slight change in the numbers given in Eq. (16) leads to what one terms as regular motion. The phenomenon of chaos, first hinted by H. Poincare about 110 years ago, has now become the talk of the day through this example in the context of non-linear systems in general.

(b) Ermakov Systems: At the end of eighteen-th century Russian Mathematician V.P. Ermakov suggested the existence of an integral invariant for a pair of coupled differential equations. For dynamical systems, however, these results were employed later by Lewis, Lewis and Riesenfeld and in a series of papers by Ray and Reid. In the context of dynamical systems, particularly for the TD harmonic oscillator this integral invariant is termed as ‘Ermakov’ (or ‘Lewis’) invariant. As a matter of fact, the work of Ermakov suggested a method to find this dynamical invariant for a system, if it exists. The dynamical invariant is a phase space function \( I(x,p,t) \), which satisfies
\[ \frac{dI}{dt} = \frac{\partial I}{\partial t} + [I,H]_{PB} = 0 \] …(18)
where \( H(x,p,t) \) is the Hamiltonian of the system and \([.]._{PB} \) is the Poisson bracket. In what follows, we overview the recent developments made in the studies of Ermakov invariants for the TD oscillator problem.

Note that the Hamiltonian corresponding to system given in Eq. (4) (or to system given in Eq. (6) for that matter), viz.,
\[ H(x,p,t) = \frac{p^2}{2m} + \frac{1}{2} m \omega^2(t) x^2, \] …(19)
is not a constant of motion as it depends explicitly on time. Therefore, one looks for other constant of motion \( I(x,p,t) \) and it turns out to be (for various methods of its derivation and interpretations see Ref. (13))
\[ I(x,p,t) = c^2 (x/\rho)^2 + (\dot{x}\rho - x\dot{\rho})^2 \] …(20)
where \( \rho(t) \) is an auxiliary function of \( t \) satisfying a non-linear equation (called Milne’s equation) of the form
\[ \dot{\rho} + \omega^2(t) \rho = c^2 / \rho^3(t) \] …(21)

In the literature, while there are now several methods available to derive the result given in Eq. (20), in the method of Ermakov, however, one needs Eq. (21) to be known in advance. In that case, the elimination of \( \omega^2(t) \) from Eqs. (4) and (21) yields
\[ (\rho \ddot{x} - \dot{\rho} x) = -c^2 x / \rho^3 \]
which, after multiplying by \( 2(\rho \ddot{x} - \dot{\rho} x) \), can be expressed as
\[ \frac{d}{dt} \left[ (\rho \ddot{x} - \dot{\rho} x)^2 + c^2 \left( \frac{x}{\rho} \right)^2 \right] = 0 \]
implying immediately the form of the invariant in Eq. (20) as the result of integration of this expression. Eqs (4), (20) and (21) constitute an Ermakov system. For several generalizations (or departures) of the TD harmonic oscillator given in Eq. (4) we refer to our recent works.

The dynamical invariant given in Eq. (20) is independent of the nature of the function \( \omega(t) \). This fact can be attributed to what is known as ‘non-linear superposition principle’ and is inherent in the structures given in Eqs. (4), (20) and (21). To this effect there exist a theorem in the literature according to which if \( x_1(t) \) and \( x_2(t) \) are the linearly independent solutions of Eq. (4) with the Wronskian \( W = (x_1 \dot{x}_2 - \dot{x}_1 x_2) \), then the general solution of Eq. (21) is given by
\[ \rho(t) = \left[ A x_1^2 + 2C x_1 x_2 + B x_2^2 \right]^{1/2} \]
where \( A,B,C \) are connected through \( AB-C^2 = \frac{c^2}{W^2} \). In what follows, we extend these concepts in
the quantum domain mainly on the basis of structural analogy, but with a bit of physics content in it.

(c) Quantum Analogue of Ermakov Systems: With a view to having further insight into the foundational aspect of Schrödinger quantum mechanics (SQM) efforts\textsuperscript{14-16} have been there in the literature to study the quantum analogue of Ermakov systems. As a result, several interesting features like a new quantization condition (called\textsuperscript{14} Milne’s quantization condition which turns out to be better than the existing WKB one) and the possibility\textsuperscript{17} of a geometric constraint in the SQM, as the manifestation of a fundamental phase in the quantum wavefunction, have been found.

The quantum analogue of the Ermakov system (cf. Eqs (4), (20) and (21) describing the case of a TD harmonic oscillator) can be derived by identifying $t \rightarrow x$, $\omega^2(t) \rightarrow k^2(x)$, $x(t) \rightarrow \psi(x)$ and thus leading to

(i) the stationary-state Schrödinger equation from Eq. (4) as

$$\psi''(x) + k^2(x)\psi(x) = 0 \quad \ldots(23)$$

(ii) a ‘space invariant’ (in place of Ermakov invariant) from Eq. (20) as

$$K = c^2 \left( \frac{\psi}{u} \right)^2 + \left( \psi'u - \psi u' \right)^2 \quad \ldots(24)$$

and (iii) the Milne-type non-linear differential equation (cf. Eq. (21)) satisfied by the function $u(x)$,

$$u''(x) + k^2(x)u(x) = c^2 / u^3(x) \quad \ldots(25)$$

where the primes denote the derivatives with respect to $x$.

Alternatively, one can make an ansatz for the solution of Eq. (23) as $\psi(x) = N u(x) \exp( i f(x) )$, where $f(x)$ is an arbitrary phase function. Using this form of $\psi(x)$ in Eq. (23) and then equating the real and imaginary parts of the resultant expression to zero yields

$$u'' + \left[ k^2(x) - f'^2 \right] u = 0; \quad \psi'' + 2 f' u' = 0.$$ 

While the second of these equations immediately provides an expression for the phase function $f(x)$ as $f(x) = c \int_0^x dx' / u^2(x')$, the use of this result in the first equation however gives the Milne-type Eq. (25). Further, by eliminating $k^2(x)$ from Eqs. (23) and (25) as before, one obtains the space invariant in Eq. (24) which in turn satisfies $(dK / dx) = 0$. Also, a result analogous to Eq. (22) in this case would suggest\textsuperscript{17} the existence of a non-linear superposition principle in the SQM. This confirms the existence of a space invariant $K$ in the SQM for a particular class of solutions of the Schrödinger equation.

Normally, one sets the mathematical equations by modelling the physical phenomenon and then looks for their solutions. Here, however, the situation is different and it is rather unique in its own way. In fact, the identification of a mathematical equation like Eq. (4) with the Schrödinger Eq. (23), as a result of structural analogy, becomes a reality here.

4 Pendula in Physical Sciences

The different branches in physical sciences can broadly be classified (Ref. (2), Chap. 4) as (A) mathematical disciplines (which include classical, quantum, statistical and stochastic mechanics and also classical and quantum field theories), (B) conceptual disciplines (consisting of condensed matter, molecular, atomic, nuclear, quark and elementary particle physics, and astrophysics), (C) conventional disciplines (like heat, sound, optics, electricity and magnetism), and (D) engineering disciplines. The fact is that the role of pendula/oscillators appears without failure in each of these disciplines and sub-disciplines. Here, we briefly pinpoint some phenomena whose theoretical understanding has required the concept of an oscillator. In fact, an oscillator model of the concerned phenomenon provides its understanding in the lowest-order only and then arise corrections in the higher-orders.

4.1 Mathematical disciplines

The abstraction stage (cf. Ref. (2), Chap. 5) of these studies at the level of Lagrangian, Hamiltonian or Hamilton-Jacobi formulations of classical mechanics is noted here. The pendulum with its variants has provided the testing ground for these methodologies of classical mechanics. For that matter, the motion of an object in a variety of practical problems in dynamics, statics and in mechanical engineering is frequently approximated by that of an oscillator or a pendulum.
In quantum mechanics, it is well known that the Schrödinger equation for the one-dimensional oscillator potential expressed as

\[ -\frac{\hbar^2}{2m} \psi''(x) + \frac{1}{2} m \omega^2 x^2 \psi(x) = E \psi(x), \quad \cdots(26) \]

admits the solutions in terms of the Hermite polynomials. The knowledge of eigenvalues and eigenfunctions thus obtained has lend support to various concepts at the foundation level of quantum mechanics. Interestingly, the solution of the Schrödinger equation for the three-dimensional (isotropic) oscillator potential, when obtained in the polar coordinates, has provided a basis for understanding the structure of many nuclei via shell model, of course after accounting for some corrections like the ones arising from the spin-orbit coupling of the individual nucleons.

While the role of harmonic forces is well known in computing various thermodynamical quantities in statistical mechanics, the harmonic potential, on the other hand, has also been the inspiring case in the development of stochastic mechanics. Also, it is only for the harmonic potential that one can easily write the complex version of the creation and annihilation operators in the form

\[ a_k = \frac{1}{\sqrt{2}} \left( x_k + i p_k \right), \quad a_k^\dagger = \frac{1}{\sqrt{2}} \left( x_k - i p_k \right) \]

and thereby develop the field theory by defining a quantity called field variable, more in analogy with the mechanics of continuous media (formation of normal modes). Polynomial field theories (particularly the $\phi^4$-one in additions to the harmonic $\phi^2$-term) have opened great vistas in nature in recent times.

4.2 Conceptual disciplines

To explain thermal and other properties of solids, one uses a spring-like model for the lattice structure of solids. According to this picture, atoms are considered to be connected by springs and undergo vibrations (harmonic or otherwise) as the temperature of the solid increases. In the context of nuclear physics, a mention of the shell model, which describes the structure of the nucleus and based on the harmonic oscillator potential is already made.

In some quark models of the nucleon (baryons in general), while the three valence quarks are considered to be bound by pair-wise harmonic forces, on the other hand, the confining piece in the quark-antiquark potential has also been considered as of harmonic type in some of the theories of quark confinement. In physical terms, these models while have successfully explained various properties of the concerned system, in mathematical terms, however, the main advantage of using single-particle harmonic potential in a many-body system lies in the fact that it only for this potential, the separation of relative and center of mass motions is easy and this simplifies the computation of the problem.

Next, in astrophysics, we mention the example of a rotating bucket of Newton which offers the case of a TD harmonic oscillator. In fact, the combination of the cosmological principle with the Einstein field equations supply the answer to some of the questions posed in the theories of Newton and Mach. For this purpose, a physical system $S$, like the solar system whose size is much less than the cosmological scale factor, $R(t)$, is considered as a rotating bucket filled with a fluid of time-varying density $\rho(t)$. Such a system is considered to be placed in a spherical cavity, cut out of the expanding universe, and so long as the size of this cavity is much less than $R(t)$, one can consider this cavity to be empty apart from the system $S$. Thus, by considering the universe as consisting of Newtonian gas in a state of everywhere-uniform expansion, the trajectory of any such gas particle of mass $m$ is described by

\[ x(t) = x(t_0) R(t) / R(t_0), \]

The gravitational potential energy $V$ of the particle, in this case, can be written as

\[ V(t) = -\frac{4\pi}{3} m G \rho(t) R^2(t) / R^2(t_0) \]

and for the kinetic energy, one writes

\[ T(t) = \frac{1}{2} m \left( |\dot{x}(t)|^2 - \frac{1}{2} m |x(t_0)|^2 \dot{R}^2(t) / R^2(t_0) \right) \]

leading to the total energy $E$ as

\[ E = T(t) + V(t) = \frac{1}{2} m \left( |x(t_0)|^2 / R^2(t_0) \right) \]
The equation of motion, then turns out to be
\[ \ddot{R}(t) + \frac{8\pi G}{3} \rho(t) R(t) = 0. \]
which is similar to Eq. (4) in mathematical terms.

4.3 Conventional and engineering disciplines
Engineering disciplines indeed are the outcome of the applicational aspects of the basic ideas developed in different branches of pure science. In the same way, the basic ideas of physics, as and when find applications in the day-to-day life, they become the part of engineering disciplines. Accordingly, the technologies are developed to have more and more applications of the idea and to make the life comfortable there of. A variety of equipment and machinery in the engineering laboratory, particularly in mechanical engineering, can be found which work on the principle of pendulum.

In conventional disciplines like heat, optics, sound, electricity and magnetism, the principle of pendulum is very widely used in different contexts. Some typical cases where this principle is used, mostly on the basis of structural analogy, are listed in Table 1. The harmonic oscillator plays very wide role in various conventional and engineering disciplines (Table 1).

5 Pendulum Experiment and the Role of Consciousness in Physical Theories
In this section we continue to discuss the pendulum experiment as a device which can possibly provide some clue for the role of consciousness in physical theories. For this purpose, we briefly highlight here three distinct situations, namely (i) when the bob alone is the source of consciousness (conscious pendulum\textsuperscript{22}), (ii) when the support alone is the source of consciousness (Chevreul pendulum\textsuperscript{2,23}) and (iii) when both the support and the bob are the sources of consciousness. In all these situations one can further consider the cases when the time-measuring apparatus is (a) an inanimate object such as an automatic electronic device, or (b) an animate object, e.g., a student is directly measuring the time period.

The ‘physics pendulum’ experiment corresponds to a very specific case in this scenario and in that there is no scope of consciousness or subjectivity whatsoever arising from the human component. The case when both the support and the bob are the sources of consciousness is as such rather involved, particularly when the time-measuring device is also a human being. The physical laws, in this case, completely fail to predict anything definitely. The essences of life of human being, namely mind, intellect, ego in addition to the biological body and the sense organs responsible for the knowledge and action, start playing an important role in any kind of direct measurement. The patomic model\textsuperscript{24} of the human being, as for the case of conscious pendulum (cf. Section 5.1), can again be helpful in this case. In fact, an explanation of some of the observations of Chevreul (cf. Section 5.2) with his hand-held pendulum can be sought within this framework. Here, however, we remark only on the first two cases.

5.1 The case of a conscious bob
Earlier\textsuperscript{2,22} we have discussed the case of a conscious pendulum in which various types of conscious bob are considered. Again the cases of different types of time-measuring devices are considered including the animate ones. It is concluded that the time-period of the pendulum will be affected by the occurrence of the processes or forces which are not accounted for by the physical theories. It is argued that the space-time mediated interactions of physics have limited validity in this regard and are inadequate to explain some of these data. Particularly, for the explanation of these processes generated by non-physical quanta (tanmātrās) of interactions, the forces postulated are that of action-at-a-time and action-at-a-space time type in addition to the contact and the action-at-a-distance type forces of physics. Corresponding to the Planck constant $\hbar$ a new constant $g$ is introduced which will account for the effects arising from the non-physical component of the Hamiltonian.

5.2 The case of a conscious support: Chevreul’s hand-held pendulum
In 1833, the French Chemist Michel-Eugène Chevreul wrote a remarkable paper on experiments and interpretation of the “magical” or “hand-held” pendulum (here after termed as Chevreul pendulum). The Chevreul pendulum consisting of a heavy bob (say an iron ring) attached to one end of a flexible
Table 1 — Role and analogous uses of a harmonic oscillator (a pendulum), viz., Eq. (1) in conventional and engineering disciplines.

Here the time period \( T \) and the spring constant \( k \) are defined through \( T = \frac{2\pi}{\omega}, k = m\omega^2 \).

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<td>( g / \ell )</td>
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<td>rigidity of a wire using Maxwell’s needle</td>
<td>( \theta ) (angle of twist)</td>
<td>( \tau = \text{restoring couple per unit twist}, \ \theta = \text{moment of inertia.} )</td>
</tr>
<tr>
<td>6(a)</td>
<td>Searle’s apparatus (horizontal arrangement)</td>
<td>( \theta ) (angle which the vibrating rods subtend)</td>
<td>( Y = \text{Young’s modulus; } R = \text{radius and } \ell = \text{length of the wire}; \ \theta = \text{moment of inertia of the vibrating arrangement.} )</td>
</tr>
<tr>
<td>6(b)</td>
<td>(Vertical arrangement)</td>
<td>( \theta ) (angle of twist)</td>
<td>( \eta = \text{rigidity of wire, other symbols are the same as in (a).} )</td>
</tr>
<tr>
<td>7.</td>
<td>Rigidity of wire Using spring vib.</td>
<td>( x ) (linear displacement)</td>
<td>( \eta = \text{rigidity of wire, } r = \text{radius of wire of the spring, } R = \text{radius of the spring, } M = \text{suspended mass, } N = \text{no. of turns in the spring.} )</td>
</tr>
<tr>
<td>8.</td>
<td>Vibration of a beam</td>
<td>( y ) (strain produced)</td>
<td>( Y = \text{Young’s modulus; } mg = \text{force that acts at one end of the beam of length } \ell; I = \text{moment of inertia of a small cross-section of beam.} )</td>
</tr>
<tr>
<td>9.</td>
<td>Oscillations of a Gas in a cylinder</td>
<td>( y )</td>
<td>( y=\text{displacement of piston of mass } M \text{ and area of cross section } A; \ \ell = \text{length of the portion of cylin. below the piston, } P = \text{pressure exerted.} )</td>
</tr>
<tr>
<td>10.</td>
<td>Vibration of a suspended magnet</td>
<td>( \theta ) (angular displacement)</td>
<td>( M = \text{magnetic moment of the magnet, } H = \text{external magnetic-field (Horizontal component of earth’s magnetic field), } \theta = \text{M.I. of the magnet.} )</td>
</tr>
<tr>
<td>11.</td>
<td>LC-circuit</td>
<td>( Q ) (charge)</td>
<td>( L = \text{Inductance, } C = \text{capacitance.} )</td>
</tr>
</tbody>
</table>
thread and the other end of the thread is held by an “unmoving” hand, is allowed to oscillate above certain substances (say water, a piece of metal or a living being), then the presence of the latter induces the pendulum’s oscillation in spite of the fact that the arm remains immobile. From the studies of such a hand-held pendulum, some of the so called magical effects (or unusual observation from the point of view of physical laws) noted by Chevreul are (i) the state of motion of the pendulum depends on what the holder is thinking (in spite that the holder is convinced of the tightness with which he/she holds the thread), (ii) the direction of the movement is said to provide yes or no answers, that relieve holders personal responsibility for decisions and choices, and (iii) the pendulum is found to react to the presence of certain objects or liquids and even provides information, such as sex of a fetus. The pattern of oscillation also gets affected by the presence of other bodies interpolated between these substances and the bob.

Some of these observations are attributed to some mystical forces or mysterious energies for which physical theories have no explanation whatsoever. In the Open Letter written to Ampere, a top physicist of that time, Chevreul listed several other observations/situations where the role of consciousness for the complete understanding of the phenomenon cannot be ruled out. Some of these cases relate23 to (a) bird in flight, (b) billiard players, and (c) movement on a slippery surface. Note that in the Chevreul pendulum not only the holder of the thread but also other human beings present, constitute the time-measuring device.

6 Concluding Discussion

From the above survey, we find that the simple pendulum with its variants in general have been responsible for the birth of several new theories and ideas in physics in spite of its limited mathematical content. Even today, in the fore-front research problems, one first tries to model the phenomenon as far as possible in terms of harmonic oscillator, if it does not work then looks for other alternatives. This may be due to its simplicity at the conceptual and computational level. The mathematical tool of harmonic oscillator appears to be ever-green as far as the advancement of science in general and physics in particular is concerned. The pendulum experiment is also expected to throw light on the role of consciousness in physical theories. In this connection, the double-pendulum experiments can also be equally helpful. In fact, the human component plays a better and dominant role in the games of golf or cricket (the case of a batsman). The physical theories developed25 to this effect still appear to be inadequate from this point of view.

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