Unsteady laminar flow of visco-elastic fluid of second order type between two parallel plates

Ch V Ramana Murthy & S B Kulkarni
Department of Applied Sciences, Finolex Academy of Management & Technology, Ratnagiri 415 639, India

Received 26 April 2004; accepted 9 September 2004

In this paper a general solution of the unsteady state laminar flow of a visco-elastic fluid of second order type between two parallel plates has been obtained, under usual assumption of no-slip condition at the boundary. The solution is obtained not by using any of the regular methods, but by employing the concepts of Duhamel's theorem. The velocity profiles are found to be more steep and further, as time passes through, the profiles are significantly distributed. Further, as time elapses, the skin friction on the plates also increases. The applications are many and are of interest, as it approximates to the flows that are commonly encountered in technological and engineering practices.

IPC Code: F15D

A simple material can be defined as a substance for which stress can be determined with the entire knowledge of the history of the strain. Further, it has the property that all local states, with the same mass density, are intrinsically equal in response, with all observable differences in response being due to definite differences in the history. For any given history \( g(s) \), a retarded history \( g_\alpha(s) \) can be defined as:

\[
g_\alpha(s)=g(\alpha s); \quad 0<s<\infty, \quad 0<\alpha \leq 1 \quad \ldots (1)
\]

\( \alpha \) being termed as a retardation factor. Assuming that the stress is more sensitive to recent deformation that to the deformations at distant past, it has been proved that the theory of simple fluids yields the theory of perfect fluids as \( \alpha \to 0 \) and that of Newtonian fluid as a correction (up to the order of \( \alpha \)) to the theory of the perfect fluids. Neglecting all the terms of the order of higher than two in \( \alpha \), we have incompressible second order fluids, governed by the constitutive relation:

\[
S=-PI+\phi_1E^{(1)}+\phi_2E^{(2)}+\phi_3E^{tri}\quad \ldots (2)
\]

where

\[
E^{i}_{i,j}=U_{i,j}+U_{j,i} \quad \ldots (3)
\]

\[
E^{\alpha}_{i,j}=A_{i,j}+2U_{m,j}U_{m,i} \quad \ldots (4)
\]

In the above equations, \( S \) is the stress-tensor, \( U_i,A \) are the components of velocity and acceleration in the direction of the \( i^{th} \) coordinate \( X_i \), \( P \) is indeterminate hydrostatic pressure and the coefficients \( \phi_1,\phi_2,\phi_3 \), are material constants.

The constitutive relation for general Rivlin-Ericksen fluid also reduces to Eq. (2) when the squares and higher orders of \( E^{2} \) are neglected, the coefficients being constants while \( \phi_2=0 \), and naming \( \phi_3 \) as the coefficient of cross-viscosity. With reference to the Rivlin-Ericksen fluids, \( \phi_2 \) may be called as the coefficient of elastico-viscosity. It has been reported that a solution of poly-iso-butylene in cetane behaves as a second order fluid and that Markovitz determined the constants \( \phi_1,\phi_2,\phi_3,\phi_4,\phi_5 \).

The problem of viscous laminar flow between two parallel plates is of interest as it approximates to the flows that are commonly encountered in technological and engineering practices. Moreover, the equations of motion become remarkably simplified yielding an exact solution.

The cases of unsteady motion of a Newtonian fluid contained between concentric rotating cylinders had been examined by several research workers, while Pattabhi Ramacharyulu\(^1\) studied the case of laminar flow of a viscous liquid between two parallel plates.
Subsequently, Ramana Murthy\textsuperscript{2} studied the flow of visco-elastic fluid of second order type between two porous parallel plates. It has been found that the presence of elastico-viscous parameter increases the order of the governing equation of motion from two to three while only two natural permeable boundary conditions are available for solution. A third boundary condition was proposed by taking into effect the nature of the bounding surfaces for a meaningful solution.

Three-dimensional free convective MHD flow and heat transfer of viscous incompressible fluid through a porous medium which is bounded by a vertical infinite porous plate had been studied by Jat & Jhankal\textsuperscript{3} in which the effect of magnetic field and permeability parameters were found to have significant effects on the flow and heat transfer. Starting from the known values of flow functions for small values of the Reynolds number, the solution is extended for larger Reynolds number while studying the effects of radial outflow and inflow of a second order fluid under an enclosed rotating disc by Sharma and Biradar\textsuperscript{4}. In all the above investigations though the bounding surfaces are of different geometries, the effect of elastico-viscous nature of the fluid under consideration on the flow variables has not been taken into account. Further, the nature of the field variables are examined and obtained by employing the classical methods and sometimes by finite difference approximations.

The aim of the present investigation is to examine the general problem of unsteady flow between two infinitely long parallel plates under the usual assumption of no-slip on the solid boundary. This problem has been studied by several investigators for over a decade. However, in the present case, instead of obtaining the solution in traditional way, a novel concept of Duhamel's theorem was applied to analyse the flow variables. The general unsteady state problem is divided into three classes of physical problems, each corresponding to a set of conditions at the boundaries. These relate to the specification of (i) fluid velocity on both boundaries, (ii) fluid velocity on one boundary and shear stress on the second boundary, (iii) shear stress on both boundaries as functions of time, in general, it is noticed that the first two classes of problems are well posed, while third class amounts to the specification of velocity gradient at both the boundaries. Such simplification is incompatible with the momentum conservation and constitutes a problem which is not well set-up and no solution is found for this case.

These three classes of boundary conditions, when expressed mathematically, are found to be special cases of one more general set of boundary conditions. Each of the original sets corresponds to a particular choice for the constants appearing in the more general set.

**Formulation of the Problem**

A set of rectangular cartesian co-ordinates \((X, Y, Z)\) with the axis of the \(Z\) along one of the two plates separated by a distance \(L\) and the axis of the \(Y\) perpendicular to the plane of the plates, the plates being \(Y = 0\) and \(Y = L\) in the dimensional form.

The two plates are assumed to be infinite so that all the physical quantities are independent of \(Z\). Now the velocity components characterizing the rectilinear flow between the plates are given by \(\{0, U(Y, T)\}\) in the \(Y\) and \(Z\) directions.

The equation of motion is given by:

\[
\rho \ddot{U} - \frac{\partial P}{\partial X} + \frac{\phi_1}{\phi} \frac{\partial^2 U}{\partial Y^2} + \phi_2 \frac{\partial}{\partial T} \left( \frac{\partial^2 U}{\partial Y^2} \right) = 0
\]

Introducing the non-dimensional parameters as:

\[
U_i = \frac{\phi u}{\rho L}, \quad T = \frac{\rho L^2 t}{\phi}, \quad \phi_2 = \rho L^3 \beta, \quad P = \frac{\phi_1^2 P}{\rho L^2}
\]

\[
X_i = Lx, \quad Y_j = Ly, \quad \phi_3 = \rho L^3 \nu_e
\]

\[
A = \frac{\phi_1^2 a_i}{\rho^2 L^3}, \quad S_{i,j} = \frac{\phi_1^3 s_{i,j}}{\rho L^5}, \quad E_{i,j}^{(1)} = \frac{\phi_1^5 \epsilon_{i,j}^{(1)}}{\rho L^5}, \quad E_{i,j}^{(2)} = \frac{\phi_1^5 \epsilon_{i,j}^{(2)}}{\rho^2 L^3}
\]

The equation of motion in the non-dimensional form is:

\[
\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \beta \frac{\partial}{\partial t} \left( \frac{\partial^2 u}{\partial y^2} \right)
\]

with the initial condition:

\[
u(y,0) = f(y)
\]

together with the boundary conditions:
\[ \alpha_1 u(0,t) + \alpha_2 \frac{\partial u(0,t)}{\partial y} = \psi_1(t), \quad t \geq 0 \]  \quad (8)

\[ \alpha_1 u(1,t) + \alpha_4 \frac{\partial u(1,t)}{\partial y} = \psi_2(t), \quad t \geq 0 \]  \quad (9)

where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are constants and \( \psi_1, \psi_2 \) are prescribed functions of the time parameter \( t \) while \( f(y) \) is a prescribed function of \( y \). So as to maintain the generality of the problem, it is assumed that the pressure gradient \( \frac{-\partial p}{\partial x} = \phi(t) \).

### Solution of the Problem

The boundary value problem formulated above involves a non-homogenous differential equation (6) with two varying boundary conditions and a non-zero initial condition. The linearity of the governing Eq. (6), the boundary and the initial conditions allow us a linear transformation of type:

\[ u(y,t) = u_1(y,t) + u_2(y,t) + u_3(y,t) + u_4(y,t) \]  \quad (10)

where the functions \( u_1(y,t) \) and \( u_2(y,t) \) are associated with respective line varying boundary conditions on \( y=0 \) and \( y=1 \), the function \( u_3(y,t) \) with the viscous decay of the initial velocity distribution with the exclusion of the pressure term in the Eq. (5) and \( u_4(y,t) \) corresponding to the sudden application and subsequent maintenance of the pressure gradient \( \phi(t) \), i.e., the functions \( u_1(y,t), u_2(y,t), u_3(y,t) \) and \( u_4(y,t) \) are to be determined.

#### Determination of \( u_1(y,t) \) and \( u_2(y,t) \)

The auxiliary problem which determines \( F_1(y,t) \) is:

\[ \frac{\partial^2 F_1}{\partial t^2} = \beta \frac{\partial^2 F_1}{\partial y^2} + \beta \frac{\partial}{\partial t} \left( \frac{\partial^2 F_1}{\partial y^2} \right) \]  \quad (11)

The initial condition: \( F_1(y,0) = 0 \) together with the boundary conditions:

\[ \alpha_1 F_1(0,t) + \alpha_2 \frac{\partial F_1(0,t)}{\partial y} = 0 \]  \quad (13)

Letting \( F_1(y,t) = F_s(y)+F_u(y,t) \) where \( F_s(y), F_u(y,t) \) are steady and unsteady state solutions. we get,

\[ \frac{\partial^2 F_s}{\partial y^2} = 0, \quad \text{at} \quad t = 0 \]  \quad (14)

subject to:

\[ \alpha_1 F_s(0)+\alpha_2 \frac{\partial F_s(0)}{\partial y} = 1 \]  \quad (15)

\[ \alpha_1 F_s(1)+\alpha_4 \frac{\partial F_s(1)}{\partial y} = 0 \]  \quad (16)

For which steady state solution is

\[ F_s(y) = A y + B_1 \]  \quad (17)

where

\[ A = \frac{[1+\alpha_1(\alpha_3+\alpha_4)]}{\alpha_4(\alpha_2\alpha_3-\alpha_4\alpha_1-\alpha_4\alpha_1)} \]  \quad (18)

and

\[ B_1 = -\frac{(\alpha_3+\alpha_4)}{(\alpha_2\alpha_3-\alpha_4\alpha_1-\alpha_4\alpha_1)} \]  \quad (19)

The unsteady state motion is governed by:

\[ \frac{\partial F_u}{\partial t} = \beta \frac{\partial}{\partial t} \left( \frac{\partial^2 F_u}{\partial y^2} \right) \]  \quad (20)

together with the boundary conditions:

\[ \alpha_1 F_u(0,t) + \alpha_2 \frac{\partial F_u(0,t)}{\partial y} = 0 \]  \quad (21)

\[ \alpha_1 F_u(1,t) + \alpha_4 \frac{\partial F_u(1,t)}{\partial y} = 0 \]  \quad (22)

\[ F_s(y) = -F_u(y,0) \]  \quad (23)
Assuming \( F_u = e^{\frac{-i\lambda_j^2}{\beta}} [A \cos y + B \sin y] \) as the trial solution

\[ F_u(y,t) = \sum_{j=1}^{\infty} C_j p_j(y) e^{\frac{-i\lambda_j^2}{\beta}} \quad ... (24) \]

where the eigen functions are given by

\[ p_j(y) = a_j \lambda_j \cos \lambda_j y - a_j \sin \lambda_j y \quad ... (25) \]

and the eigen values \( \lambda_j \) are the roots of

\[ \tan \lambda = \frac{(\alpha_3 \lambda_j - \alpha_4 \lambda_j)}{(\alpha_3 \lambda_j + a_2 \alpha_4 \lambda_j^2)} \quad ... (26) \]

It is noticed that the sequence \([ p_j(y) \] is orthogonal in \([0,1]\) with in the norm

\[ \| p_j(y) \|^2 = \frac{1}{2} \left( (\alpha_3^2 + \alpha_4^2 \lambda_j^2) + (\alpha_3 \lambda_j - \alpha_4 \lambda_j)(\alpha_3 \lambda_j + \alpha_4 \lambda_j^2) \right) \quad ... \]}

\[ \left( \alpha_3^2 + \alpha_4^2 \lambda_j^2 \right) \]

The constants \( C_j \) are determined from the initial condition of the equation

\[ -F_u(y,t) = \sum_{j=1}^{\infty} C_j p_j(y) \quad ... (28) \]

which yields

\[ C_j = \frac{1}{\lambda_j \| p_j(y) \|^2} \quad ... (29) \]

Hence

\[ F_1(y,t) = F_s(y) + F_u(y,t) \quad ... (30) \]

\[ F_1(y,t) = A_1 y + B_1 + \sum_{j=1}^{\infty} C_j p_j(y) e^{\frac{-i\lambda_j^2}{\beta}} \quad ... (31) \]

Employing Duhamel's theorem

\[ u_1(y,t) = \int_0^t \psi_1(\tau) \frac{\partial F_1(y,t-\tau)}{\partial t} \, d\tau \quad ... (32) \]

\[ u_1(y,t) = -\frac{1}{1 + \beta} \sum_{j=1}^{\infty} C_j p_j(y) e^{\frac{-i\lambda_j^2}{\beta}} \int_0^t \psi_1(\tau) e^{\frac{-i\lambda_j^2}{\beta}} \, d\tau \quad ... (33) \]

Proceeding for \( u_2(y,t) \) in the same lines as that for \( u_1(y,t) \)

\[ u_2(y,t) = -\frac{1}{1 + \beta} \sum_{j=1}^{\infty} D_j p_j(y) e^{\frac{-i\lambda_j^2}{\beta}} \int_0^t \psi_2(\tau) e^{\frac{-i\lambda_j^2}{\beta}} \, d\tau \quad ... (34) \]

where

\[ D_j = \sqrt{\frac{\alpha_3^2 + \alpha_4^2 \lambda_j^2}{\alpha_3^2 + \alpha_4^2 \lambda_j^2}} \quad ... (35) \]

**Determination of \( u_3(y,t) \)**

It is seen that \( u_3(y,t) \) can be written as

\[ u_3(y,t) = \sum_{j=1}^{\infty} E_j p_j(y) e^{\frac{-i\lambda_j^2}{\beta}} \quad ... (36) \]

where the constants \( E_j \) can be obtained by using initial condition.

\[ E_j = \frac{\int f(y) p_j(y) \, dy}{\| p_j(y) \|^2} \quad ... (37) \]

**Determination of \( u_4(y,t) \)**

\[ \frac{\partial F_s}{\partial t} = \frac{\partial p}{\partial x} + \frac{\partial^2 F_s}{\partial y^2} + p \frac{\partial}{\partial t} \left( \frac{\partial^2 F_s}{\partial y^2} \right) \quad ... (38) \]

and letting the solution as: \( F_s(y,t) = F_s(y) + F_u(y,t) \) where \( F_s(y), F_u(y,t) \) are steady and unsteady state solutions. The boundary and initial conditions will now be

\[ \alpha_1 F_s(0) + \alpha_2 \frac{\partial F_s(0)}{\partial y} = 1 \quad ... (39) \]
\[ \alpha_s F_s(l) + \alpha_e \frac{\partial F_s(l)}{\partial y} = 0 \]  \hspace{1cm} \text{(40)}

and

\[ F_s(y) = -F_s(y,0) \]  \hspace{1cm} \text{(41)}

for which the steady state solution is

\[ F_s(y) = \frac{1}{1 + \beta} \left[ \frac{y^2}{2} + A_1 y + B_1 \right] \]  \hspace{1cm} \text{(42)}

where

\[ A_1 = \frac{2\alpha_s - \alpha_s \alpha_e - 2\alpha_e \alpha_s}{2\alpha_e \alpha_s + 2\alpha_s - 2\alpha_e \alpha_s} \]  \hspace{1cm} \text{(43)}

and

\[ B_1 = \frac{1 + \alpha_s (2\alpha_s - \alpha_s \alpha_e - 2\alpha_e \alpha_s)}{-\alpha_s} \]  \hspace{1cm} \text{(44)}

The unsteady state solution is given by:

\[ F_u(y,t) = \sum_{j=1}^{\infty} F_j p_j(y) e^{-\frac{\alpha_j^2}{y^2}} \]  \hspace{1cm} \text{(45)}

where the constants \( F_j \) can be obtained by using initial condition.

\[ F_j = \frac{\int f(y) F_j(y,0) p_j(y) dy}{\| p_j(y) \|^2} \]  \hspace{1cm} \text{(46)}

Hence,

\[ u(y,t) = \left( \frac{y^2}{2} + A_1 y + B_1 \right) + \sum_{j=1}^{\infty} F_j p_j(y) e^{-\frac{\alpha_j^2}{y^2}} \]  \hspace{1cm} \text{(47)}

where \( A_1, B_1 \) are given the relations (43) and (44) respectively.

---

**Fig. 1**—Velocity versus plates separation for \( t = 0.3 \)
Employing Duhamel's theorem, then

\[ u(y,t) = \frac{1}{1+\beta} \sum_{j=1}^{\infty} \left( \alpha_j C_j + \alpha_2 D_j \right) p_j(y) e^{-\gamma_j^2 \frac{t^2}{1+\beta}} \int_{-\infty}^{t} \phi(\tau) e^{\frac{1}{1+\beta} \gamma_j^2 \tau} d\tau \]

... (48)

Therefore, the complete solution of the velocity field satisfying the conditions (7-9) are given by:

\[ u(y,t) = \sum_{j=1}^{\infty} p_j(y) e^{\frac{-\gamma_j^2 t^2}{1+\beta}} \]

\[ \left[ E_j - \int_{0}^{t} \frac{1}{1+\beta} \gamma_j^2 (C_j \psi_j(\tau) + D_j \psi_2(\tau)) e^{\frac{-\gamma_j^2 \tau}{1+\beta}} d\tau \right] e^{\frac{-\gamma_j^2 t^2}{1+\beta}} + (\alpha_j C_j + \alpha_2 D_j) \phi(\tau) \]

... (49)

Skin-friction on lower and upper plates are given by \( \left( \frac{\partial u}{\partial y} \right)_{y=0} \) and \( \left( \frac{\partial u}{\partial y} \right)_{y=1} \) respectively.

**Results and Discussion**

The velocity profiles for various values of \( t \) and \( \beta \) are presented graphically in Figs 1 and 2. It is observed that as the elastico-viscosity parameter \( \beta \) increases, the velocity profiles are found to be more steep and further, as \( t \) increases, the profiles are significantly distributed.

The skin friction on lower and upper plates for different values of elastico-viscosity parameters \( \beta \) and \( t \) are presented graphically in Fig. 3. It is observed that as \( t \) increases, the skin friction on the plates also increases.

Further, as the elastico-viscosity parameter \( \beta \rightarrow 0 \), the results coincide with that of a Newtonian fluid.

**Conclusions**

From this study, it is found that as the elastico viscosity of the fluid increases while retaining the material constants the velocity profiles are more parabolic in nature. Due to the sudden application of pressure gradient, the intra molecular forces tends to pull the neighbouring cluster of fluid element as a result of which the fluid velocity at the bottom plate is high and gradually reduces. At a shorter duration...
(i.e as \( t \) reduces), the velocity profiles are found to be more parabolic. This is due to the fact that immediately after the application of the pressure gradient, the velocity of the fluid near the bounding surfaces is high and as time passes, the intra molecular forces reduces which is in accordance with the physical observable phenomena. It is also found that for a constant-elastico viscosity of the fluid, the profiles are more steep with increase of time.

Acknowledgements

The authors express their sincere thanks to Dr R A Mashelkar, Director General of CSIR for his critical comments and suggestions and to Dr K L Asanare, Director, Finolex Academy of Management and Technology, Ratnagiri, for providing necessary computational facilities and constant encouragement.

References