Design of LQR controller for active suspension system

M Senthil Kumar & S Vijayarangan
Department of Mechanical Engineering, PSG College of Technology, Coimbatore 641 004, India

Received 13 June 2005; accepted 16 September 2005

The present paper aims at developing an active suspension for a passenger car by designing a controller using linear quadratic optimal control theory. In this work, two different control approaches are proposed, viz., conventional method (CM) and acceleration dependent method (ADM). A quarter car model with 3 degrees-of-freedom has been considered for the analysis. The performance of the active suspension system with two control approaches has been compared with that of passive one. It is concluded that active suspension system has a better potential to improve both the ride comfort and road holding, since the RMS (Root Mean Square) passenger acceleration has been reduced by 54.23% for active CM system and by 93.88% for active ADM system compared to passive one, and suspension travel has also reduced to about 37.5%.

IPC Code: F16

The main aim of suspension system is to isolate a vehicle body from road irregularities in order to maximize passenger ride comfort and retain continuous road-wheel contact in order to provide road holding\(^1\,^2\). Traditionally automotive suspension designs have been a compromise among the three conflicting criteria of road holding, load carrying and passenger comfort, which are also called as design goals\(^3\). Good ride comfort requires a soft suspension, whereas insensitivity to applied loads requires stiff suspension. Good handling requires a suspension setting somewhere between the two. Demands for better ride comfort and controllability of road vehicles like passenger cars have motivated to develop a new type of suspension systems like active suspensions. The electronically controlled suspension systems can potentially improve the ride comfort as well as the road holding of the vehicle\(^5\). Road holding relates contact forces of the tyres and road surface. In passive system, the parameters are fixed, being chosen to achieve a certain level of compromise between road holding, load carrying and comfort. An active suspension system, on the other hand, has the capability to adjust itself continuously to changing road conditions resulting in a better set of design trade-offs compared to passive suspension. Active suspension systems employ pneumatic or hydraulic actuators for additional energy. The actuator is secured in parallel with a spring and shock absorber\(^5\,^11\). Active suspension requires sensors to be located at different points of the vehicle to measure the motions of the body. This information is fed as input for the controller in order to provide exact amount of force required through the actuator.

Mathematical Modelling

In this work, a quarter car model with three degrees-of-freedom is considered. It not only leads to simplified analysis but also represents most of the features of the full model. The model consists of passenger seat and sprung mass that is supported on springs and unsprung mass which refers to the mass of wheel assembly. The tyre has been replaced with its equivalent stiffness and tyre damping is neglected. The suspension, tyre and passenger seat are modelled by linear springs in parallel with dampers.

Passive suspension

For passive suspension shown in Fig. 1, using the Newton’s second law of motion and free-body diagram concept, the equations of motion are derived.

\[
\begin{align*}
0 = m_p \ddot{x}_p + k_p (x_p - x_s) + c_p (\dot{x}_p - \dot{x}_s) = 0 & \quad \cdots (1) \\
0 = m_x \ddot{x}_x + k_p (x_x - x_p) + c_p (\dot{x}_x - \dot{x}_p) + k_i (x_i - x_m) + c_i (\dot{x}_i - \dot{x}_m) = 0 & \quad \cdots (2) \\
0 = m_{xx} \ddot{x}_{xx} + k_i (x_{xx} - x_x) + c_i (\dot{x}_{xx} - \dot{x}_x) + k_i (x_{xx} - r) = 0 & \quad \cdots (3)
\end{align*}
\]

Using following notations as given by Lin and Kanellakopoulos\(^5\),
the equations of motion [Eqs (1-3)] can be written in state variable form where \( x_1, x_2, x_3, x_4, x_5 \) and \( x_6 \) are taken as the states, (the dot over the letters indicates differentiation with respect to time). Therefore,

\[
\dot{x}_2 = \dot{x}_p = \frac{-1}{m_p}[k_p(x_1 - x_1) + c_p(x_2 - x_4)] \quad \ldots (4)
\]

Writing these equations in state space representation form \(^7\), i.e., \( \dot{x} = Ax + Gr \), we get,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-k_p & -c_p & k_p & 0 & 0 & 0 \\
0 & m_p & m_p & 0 & 0 & 0 \\
-k_p & -c_p & (k_s + k_p) & (c_s + c_p) & k_s & c_s \\
0 & m_s & m_s & 0 & 0 & 0 \\
0 & 0 & k_s & c_s & -\left(\frac{k_s + k_i}{m_{us}}\right) & -\frac{c_s}{m_{us}} \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} r
\]

where \( A = \ldots \) and \( G = \ldots \)

Active suspension

Active suspension system has a hydraulic actuator in addition to the passive elements. The hydraulic actuator is located parallel to the suspension spring and shock absorber. For active suspension shown in Fig. 2, using the Newton’s second law of motion and free-body diagram concept, the equations of motion are derived.

\[
m_p\ddot{x}_p + k_p(x_x - x_p) + c_p(\dot{x}_x - \dot{x}_p) = 0 \quad \ldots (8)
\]

Using following notations as given in Eq. (3),

\[
x_1 = x_p, \ x_2 = \dot{x}_p, \ x_3 = x_x, \ x_4 = \dot{x}_x,
\]

\[
x_5 = x_{us}, \ x_6 = \dot{x}_{us} \quad \ldots (8)
\]
we get,
\[ \dot{x}_2 = \ddot{x}_p = -\frac{1}{m_p}[k_p(x_1-x_3)+c_p(x_2-x_4)] \]  \hspace{1cm} \cdots (11)

\[ \dot{x}_4 = \ddot{x}_s = -\frac{1}{m_s}[k_p(x_3-x_1)+c_p(x_3-x_2) + k_s(x_3-x_3)+c_p(x_4-x_2)-f_a] \]  \hspace{1cm} \cdots (12)

Writing these equations in state space representation form, i.e, \( \dot{x} = Ax + Bu + Gr \)  \hspace{1cm} \cdots (14)

we get,
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{k_p}{m_p} & -\frac{c_p}{m_p} & \frac{k_p}{m_p} & \frac{c_p}{m_p} & 0 & 0 \\
0 & \frac{c_p}{m_s} & 0 & \frac{c_p}{m_s} & 0 & 0 \\
\frac{k_s}{m_s} & \frac{c_p}{m_s} & \frac{k_s}{m_s} + \frac{c_p}{m_s} & \frac{k_s}{m_s} & 0 & 0 \\
0 & 0 & 0 & \frac{c_s}{m_u} & \frac{k_s}{m_u} + \frac{c_s}{m_u} & \frac{k_s}{m_u} \\
0 & 0 & 0 & 0 & \frac{c_s}{m_u} & \frac{k_s}{m_u}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
-1
\end{bmatrix} f_a +
\begin{bmatrix}
-1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} r
\]

\hspace{1cm} \cdots (15)

where, \( B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \) and \( u = [f_a] \).
Linear Quadratic Regulator (LQR) Controller Design

The performance characteristics, which are of most interest when designing the vehicle suspension, are passenger ride comfort, suspension travel and road holding. The passenger ride comfort is related to passenger acceleration, suspension travel is related to relative distance between the unsprung mass and sprung mass and road holding is related to the tyre displacement. The controller should minimize all these quantities and hence suitable six states are selected to configure the controller. The various states considered are passenger displacement and velocity, sprung mass displacement and velocity, and tyre displacement and velocity. Hence, the state vector matrix is \( x = [x_1, x_2, x_3, x_4, x_5, x_6] \). Eq. (14) is a linear time invariant system (LTI). State feedback control for active suspension is a powerful tool for designing a controller. For controller design, it is assumed that all the states are available and could be measured exactly. This is not always the case, particularly when the active suspension system being controlled has complicated internal dynamic behaviour, which is difficult to directly measure. State estimators could be developed to estimate the state variables from a limit number of observations. A state estimator is an electrical or digital system, which models the internal dynamics of the mechanical system being controlled. The internal states of the electronic state estimator, which can of course be directly measured, will track the internal states of the suspension system, which cannot be measured directly. We would ideally use three transducers to measure the acceleration, velocity and displacement. In practice, often only a single transducer is available for example, and electronic integrators are used to derive signals proportional to velocity and displacement. However, this state estimator is not dealt in the paper. And, hydraulic dynamics of force actuator is not considered since it is assumed that the required force is assumed to be applied in between the sprung and unsprung mass. Let us consider a state variable feedback regulator;

\[
u = -Kx, \quad \ldots (16)\]

where \( K \) is the state feedback gain matrix. The performance index \( J \) represents the performance characteristic requirement as well as the controller input limitations. In this work, two different approaches are considered in order to evaluate the performance index and hence designing the optimal controller. The first approach is the conventional method (CM) in which only the system states and inputs are penalized in the performance index. But in the acceleration dependent method (ADM), special importance is given to the ride comfort by introducing passenger acceleration term in the performance index.

Conventional method (CM)

In this method, the performance index \( J \) penalizes the state variables and the inputs. Thus, it has the standard form as,

\[
J = \int_0^\infty (x^T Q x + u^T R u) dt \quad \ldots (17)
\]

where \( u = [f_a] \) and \( Q \) and \( R \) are positive definite weighting matrices. Here the passenger acceleration, which is indicator of ride comfort, is not included.

Linear optimal control theory provides the solution of Eq. (17) in the form of Eq. (16). The gain matrix \( K \) is computed from,

\[
K = R^{-1} B^T P \quad \ldots (18)
\]

where the matrix \( P \) is evaluated being the solution of the Algebraic Riccati Equation,

\[
A^T P + PA - PBR^{-1} B^T P + Q = 0 \quad \ldots (19)
\]

By substituting gain matrix \( K \) in Eq. (14), we get the controlled states in the form,

\[
\dot{x} = (A - BK)x + Gr \quad \ldots (20)
\]

Acceleration dependent method (ADM)

Acceleration dependent method is new approach giving more importance to passenger comfort by including passenger acceleration in the performance index. Suppose that the vector \( z \) represents the passenger acceleration, \( z = [\ddot{x}_p] \)

Then, the performance index could be written in the following form

\[
J = \int_0^\infty (x^T Q x + u^T R u + z^T S z) dt \quad \ldots (21)
\]

Here, \( S \) is weighting matrix, which can be suitably assumed.

Therefore, the Eq. (21) becomes
This equation could be further modified, since the passenger acceleration is linearly dependent on the state variables. Eq. (11) may be rewritten as,

\[ \dot{x}_p = v^T x \]  \hspace{1cm} \text{(23)}

where row vector \( v \) is written as,

\[ v = \frac{1}{m_p} \begin{bmatrix} -k_p & -c_p & k_p & c_p & 0 & 0 \end{bmatrix} \]  \hspace{1cm} \text{(24)}

The Eq. (22) can be written as;

\[ J = \int_{0}^{\infty} \left( x^T (Q + v^T S v) x + u^T R u \right) dt \]  \hspace{1cm} \text{(25)}

Finally, Eq. (22) could be written from Eq. (25) as,

\[ J = \int_{0}^{\infty} (x^T Q_n x + u^T R u) dt \]  \hspace{1cm} \text{(26)}

where, \( Q_n = Q + v^T S v \).

The optimal solution for Eq. (26) is found in a similar manner to that of Eq. (17).

**Results and Discussions**

To verify LQR control design for active suspension system, the resulting closed-loop system was simulated and the two control methods were compared with passive suspension. MATLAB software programs have been developed to handle the controller design and simulation for active and passive systems. Passenger ride comfort is directly related to the body acceleration, whereas suspension travel refers to the space available in between the sprung mass and unsprung mass. Wheel displacement is used as a measure of road holding ability. Both passive and active suspension systems are analyzed subjecting them to arbitrary road input, a sinusoidal function, which represents a bump on the road. The sinusoidal bump with frequency of 4 Hz has been characterized by,

\[ r = \begin{cases} a(1 - \cos 8\pi t) & 0.5 \leq t \leq 0.75 \\ 0 & \text{otherwise} \end{cases} \]  \hspace{1cm} \text{(27)}

where, \( a = 0.025 \) (road bump height 5 cm).

Using the following suspension parameters, the system response for the bump has been analysed for passive suspension and active suspension with two different methods.

- \( m_p = 60 \text{ kg} \)
- \( k_r = 16812 \text{ N/m} \)
- \( m_s = 290 \text{ kg} \)
- \( k_i = 190000 \text{ N/m} \)
- \( m_{as} = 59 \text{ kg} \)
- \( c_r = 875.6 \text{ N-s/m} \)
- \( k_p = 10507 \text{ N/m} \)
- \( c_s = 1000 \text{ N-s/m} \)

Figs 3-7 illustrate the transient response of the vehicle model under bumpy road input. From Figs 3 and 4, it is evident that passenger displacement and passenger acceleration have significantly reduced for the active system compared to passive one. Also, passenger displacement and acceleration for active...
system with acceleration dependent method (ADM) system is slightly lower than that of conventional method (CM) system.

Fig. 5 shows that the sprung mass displacement is also less for the active system than the passive one. Moreover, from Fig. 6 the tyre displacement for the active system is also less. From the suspension travel plot in Fig. 7, it can be seen that for active CM system the suspension travel has reduced by 37.5%, which shows better rattle space utilization. However, for active ADM system there is slight increase in suspension travel after the peak when compared to active CM system, but it is still lesser than the corresponding value of the passive system. This is because for active ADM controller more weightage has been given to ride comfort and since the design goals ride comfort and road holding are contradictory hence there is slight increase in suspension travel.

According to ISO 2631 standard for whole body vibration exposure limitations which states that the RMS value of acceleration denotes the total energy across the entire frequency range which is referred as a measure of comfort. Also comfort levels to vibration response in vertical direction are defined by the vibration tolerance limits, which are illustrated through the graph plotted between the RMS acceleration and frequency in Fig. 8. If the vibration exposure crosses these limits then it leads to reduction in human performance, fatigue and motion sickness.

RMS acceleration of active CM suspension and RMS acceleration of active ADM suspension are determined to be 0.7089 m/s² and 0.0948 m/s² respectively while that of passive suspension system is 1.5508 m/s². From these values, it is found that for active CM suspension the acceleration has reduced by
54.23% while for active ADM suspension the reduction in acceleration is 93.88% compared with passive suspension which guarantee better ride comfort.

From the RMS values of the passenger accelerations of active and passive suspension at a frequency of 4 Hz as seen in Fig. 8, it can be stated that, by using the vehicle with passive suspension a person can comfortably travel up to 45 min and using the vehicle with active CM suspension a person can comfortably travel up to 2 h, while using active ADM suspension a person can comfortably travel more than 8 h. This comparison clearly shows that active suspension gives better passenger comfort and that the active suspension with acceleration dependent method (ADM) gives still better passenger comfort.

Conclusions
Suspension design is a compromise among ride comfort, suspension travel and road handling. By including an active element in the suspension, it is possible to reach a better compromise than is possible using purely passive elements. The potential for improved ride comfort and better road holding using LQR controller design is examined. A 3 degrees-of-freedom quarter car model has been considered to evaluate the performance of the suspension with respect to various contradicting design goals. Two controller design approaches, namely conventional method (CM) and acceleration dependent method (ADM) have been examined for the active system.

The passenger RMS acceleration values and corresponding frequency have been used to quantify the ride comfort as per the ISO 2631 standard for whole body vibration exposure. From the simulation results, it is evident that active suspension system has a better potential to improve both the ride comfort and road holding, since the RMS passenger acceleration has been reduced by 54.23% for active CM system and by 93.88% for active ADM system compared to passive one. Also, suspension travel has reduced by about 37.5%. Furthermore, active suspension system provides improved road-holding ability. It is concluded that active suspension system using ADM gives better performance than the passive suspension system.

Acknowledgement
The authors gratefully acknowledge the support rendered by a project sanctioned by All India Council for Technical Education (AICTE), India.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_p$</td>
<td>passenger seat mass in kg</td>
</tr>
<tr>
<td>$m_s$</td>
<td>quarter car sprung body mass in kg</td>
</tr>
<tr>
<td>$m_u$</td>
<td>unsprung mass in kg</td>
</tr>
<tr>
<td>$k_p$</td>
<td>seat stiffness in N/m</td>
</tr>
<tr>
<td>$k_s$</td>
<td>spring stiffness in N/m</td>
</tr>
<tr>
<td>$k_t$</td>
<td>tyre stiffness in N/m</td>
</tr>
<tr>
<td>$c_p$</td>
<td>seat damping coefficient in N-sec/m</td>
</tr>
<tr>
<td>$c_s$</td>
<td>suspension damping coefficient in N-sec/m</td>
</tr>
<tr>
<td>$x_p$</td>
<td>passenger vertical displacement (m)</td>
</tr>
<tr>
<td>$x_s$</td>
<td>sprung mass vertical displacement (m)</td>
</tr>
<tr>
<td>$x_u$</td>
<td>unsprung mass vertical displacement (m)</td>
</tr>
<tr>
<td>$r$</td>
<td>road profile (m)</td>
</tr>
<tr>
<td>$f_a$</td>
<td>actuator force (N)</td>
</tr>
</tbody>
</table>

References