Force on vertically submerged circular thin plate in shallow water due to oblique wave

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This paper describes a solution of obliquely incident surface wave force on circular plate in shallow water. Small amplitude linear wave theory is used on vertically submerged circular thin plates under three different configurations: (1) a surface-piercing circular thin plate, (2) a submerged circular thin plate and (3) a bottom-standing circular thin plate. Finally Morison’s equation is used for the determination of wave force. The plates are submerged in water near the shore on uniformly sloping bottom. The solution method is confined in a finite domain, which contains the regions of different depth of water and the plate. Employing Laplace equation and boundary value problems in the finite domain, the desired velocity potential for small amplitude linear wave is derived by separation of variables. Horizontal wave force and moment are obtained with respect to the wave amplitude at different incident angles of wave as well as with different depth of water and different wave period. It is observed that these forces and moments are converging with the increase in wave period whereas the steepness of force and moment against wave amplitude are extremely high for small wave period.

[Keywords: Obliquely incident wave; Linear wave; Morison’s equation; Laplace equation; Wave force; Moment.]

1. Introduction

An attempt is made to introduce a new concept of floating reciprocating pump wherein kinetic energy obtainable from the ocean surface wave can be easily stored in the form of potential energy and subsequently can be used for any other purpose. The pump may float near the shore or at the deep sea. As a first step of the total design of the pump, a circular thin plate is considered which acts as a piston of the floating reciprocating pump and moves to and fro due to the action of ocean wave.

Various types of studies on wave force and moment were carried out at different time by many authors on different types of submerged structures, like vertical barrier of different geometrical shapes etc. Morison et al.¹ (1950) and Morison et al.² (1953) have proposed a simple empirical equation to estimate the velocity dependent drag force and acceleration dependent inertia force on vertical cylindrical piles. Hanssen and Torum³ (1999) experimentally studied horizontal and vertical forces as well as overturning moments due to wave acting on the tripod, using Morison’s equation. Meylan⁴ (2001) presented a variational equation for wave force on floating thin plates in his article “A variational equation for the wave forcing of floating thin plates”. Maiti and Sen⁵ (1998) developed a numerical time-simulation algorithm for analyzing highly non-linear solitary waves interacting with plane gentle and steep slopes, employing mixed Eulerian-Lagrangian method and, also found pressures and forces on impermeable wall by Bernoulli’s equation.

Sundaravadivelu et al.⁶ (1997) experimentally measured force and moments due to regular wave on an intake well using Linear Diffraction theory. Tsai and Jeng⁷ (1990) studied the forces on vertical walls due to obliquely incident waves employing Fourier Series. Mallayachari and Sundar⁸ (1994) investigated the wave pressure exerted on vertical walls due to regular and random waves, using Fourier Series approach.

Most of the other authors investigated the breaking wave forces exerted on vertical circular cylinder (e.g., Goda⁹, 1973; Sawaragi and Nohchino¹⁰, 1984; Tanimoto et al.¹¹, 1986; Apelt and Piorewiez¹², 1987). Brater et al.¹³ (1958) investigated wave force on submerged structures. Presentation of Hall¹⁴ (1958) was on a laboratory study of breaking wave forces on

In the present study, linear wave theory of small amplitude is used under three different configurations (Fig. 1). Finally Morison’s equation is used for the determination of obliquely incident wave force on vertically submerged circular thin plate. Besides these, other investigations such as moments about any point of the plate are also investigated in this paper.

In order to analyses such wave mechanics problem, value of drag coefficient ($C_D$) & inertia coefficient ($C_M$) must be known. However in practice it is extremely difficult to obtain reliable values of these. Generally they depend upon the Reynold’s number (Re) and Keulegan – Carpenter (KC). In practice, the inertia co-efficient $C_M = (1+Km)$ and thus $C_M = 2$, since $Km$ is the added mass and is unity for a cylinder; vide [Fig. 8.6, Dean and Dalrymple (2000)]. On the basis of large number of practical observations Wiegel (1957) (refer Fig. 11.8, Robert, 1964, P. 258) demonstrated that drag co-efficient $C_D = 1.6$, in the range of $RE = 20000$ to $200000$ for circular cylindrical pile of various diameters.

2. Mathematical Formulation

Consider a thin circular plate submerged vertically in shallow water under three different configurations. The plate is subjected to an obliquely incident wave which is assumed to be traveling at an angle $\theta$ measured in the clockwise direction with respect to $x$-axis. To use linear wave theory, it is assumed that the wave amplitude ($a$) is small compared to both the wave length ($L$) and the water depth ($d$). Estimates of wave forces by Morison’s equation under the assumption of the linearized theory of surface waves are done in 3D Cartesian coordinate system. The fluid occupies the region $-d \leq z \leq 0$ in $x < 0$, $-d(x) \leq z \leq 0$ in $0 < x \leq b$ except the region in the fluid which is occupied by the plate. Thin circular plate occupies the following three positions, as shown in Fig.1.

- **Type I:** $x = 0$, $-z_2 \leq z \leq 0$
- **Type II:** $x = 0$, $-z_2 \leq z \leq -z_1$
- **Type III:** $x = 0$, $-d \leq z \leq -z_1$

Consider unsteady incompressible inviscid irrotational flow without surface tension and with atmospheric pressure $P_a=0$. The system is idealized as 3D and Cartesian co-ordinates are employed in the plane $x-z$, normal to the still water surface. The $z$-axis is directed vertically upward from still water surface.

![Fig. 1—Definition sketch of three types of plate positions.](image)
and is measured positive upwards. Direction of propagation of wave represents x-axis. The plane x-y is horizontal & coincides with the still water surface. The geometry is as depicted in Fig. 1.

The governing equations for irrotational wave motion are given by the Laplace equation:

\[ \nabla^2 \phi(x,t) = 0 \quad -d \leq z \leq 0, \quad x < 0 \]  \hspace{1cm} (1)

& the associated three boundary conditions are:

(a) Dynamic boundary condition at the surface (DSBC) based on the Bernoulli’s equation, which is

\[ \rho g \eta + \rho \phi_z + P_a = 0 \quad z = 0 \]  \hspace{1cm} (2)

where \( \eta \) is the water surface elevation, \( g \) is the acceleration due to gravity, \( \rho \) is the density of water and \( P_a \) is the atmospheric pressure at the still water surface which is equal to zero i.e. \( P_a = 0 \) as assumed earlier.

(b) Kinetic boundary condition at the surface (KSBC).

\[ \phi_z = \eta, \quad z = 0 \]  \hspace{1cm} (3)

(c) Bottom boundary condition (BBC).

\[ \phi_z = 0 \quad z = -d \]  \hspace{1cm} (4)

Assume \( d(x) \) to represent the depth of water for uniformly sloping bottom in the region \( 0 \leq x \leq b \), and let the depth \( d \) be constant in the region \( -\infty \leq x \leq 0 \).

Employing Laplace eq. (1), BBC and linearized DSBC the desired velocity potential for small amplitude linear waves is derived by the method of separation of variables (Ippen, 1966 or Sorensen, 1978). An useful form of this velocity potential is

\[ \phi(x,t) = \Re \left( \frac{ag}{\omega} \times \cosh[k(z+d)] e^{i(k \cos \theta + k \sin \theta - \omega t)} \right) \]  \hspace{1cm} x < 0

\[ \omega^2 = gk \tanh(kd) \]  \hspace{1cm} (6)

As per the restrictions for linearization, the wave amplitude has to be small compared to the depth. On that basis the expressions for linear shallow water wave theory are given below by using \( \sinh(kd) \equiv kd \) and \( \cosh(kd) \equiv 1 \) for \( kd \leq 1 \).

\[ \phi(x,t) = \Re \left( \frac{ag}{\omega} \times \cosh(k(z+d)) e^{i(k \cos \theta + k \sin \theta - \omega t)} \right) \]  \hspace{1cm} x < 0

\[ \omega^2 = gdk^2 \]  \hspace{1cm} (8)

Horizontal & vertical component of water particle velocity, \( u(x,t) \) & \( w(x,t) \) are obtained from velocity potential and accelerations are obtained by differentiating \( u(x,t) \) & \( w(x,t) \) with respect to time.

3. Wave Force Calculation

In general, the force transmitted by oblique waves onto the vertically submerged circular thin plate is a function of horizontal component of velocity and acceleration, induced by waves of height \( H \) and wave period \( T \). The wave force is calculated by using Morison’s equation which is the sum of inertia force and drag force.

3.1. Horizontal wave force

\[ F_{\text{HM}} = C_M \rho V \frac{\partial u}{\partial t} + \frac{1}{2} C_D \rho A \sqrt{|u|^2} \]  \hspace{1cm} (9)

where \( C_M = \) co-efficient of mass; \( \rho = \) density of fluid; \( V = \frac{\pi}{4} D^2 l = \) volume of the fluid displaced by thin plate; \( l = \) thickness of the thin plate; \( D = \) diameter of plate; \( C_D = \) co-efficient of drag; \( A = \frac{\pi}{4} D^2 = \) projected area of plate perpendicular to the stream velocity.

According to the authors’ assumption, the plate is very thin and it is vertically submerged, so its
projected area normal to the z-axis is very small and
is assumed to be zero. Again we have from BBC, the
vertical velocity of wave \( w = 0 \). Though this velocity
gradually increases towards the water surface, it is
also very small. Therefore vertical force on the plate
is considered negligible.

To determine the horizontal force \( F_x \) on a vertically
submerged circular thin plate, the force per unit
elevation \( dz \) must be integrated over the immersed
length of the plate. The integration can be carried out
to between \( z_i \) & \( z_2 \) measured from the ocean surface, to
give an approximation of the total force. Thus

\[
F_x = \int_{z_i}^{z_2} (F_{sm}) \, dz \tag{10}
\]

4. Moment

The overturning moment about any point between
\( z_i \) to \( z_2 \) on a vertically submerged circular thin plate
due to the horizontal component of force is:

\[
M = \int_{z_i}^{z_2} (d^* - \bar{d}) \, dF_x \tag{11}
\]

where \( d^* \) = distance of centre of pressure from the free
surface,
\( \bar{d} \) = distance of centre of gravity of the plate
from the free surface.

5. Beach angle calculation

Galvin (1972) gives the following relation of
breaking wave:

\[
\frac{H_b}{\sqrt{g m_1 T^2}} = 0.068 \tag{12}
\]

where \( H_b \) = breaking wave height and \( m_1 = \tan \beta_1 \),
where \( \beta_1 \) is the beach angle at breaking. Now for
calculating the beach angle, where wave is not
breaking and does obey the small amplitude linear
wave theory, assume \( H < H_b \) and hence \( H/d < 0.88 \).
And also assume \( m < m_1 \) is the beach slope, when the
wave is not breaking. Here \( m = \tan \beta \), where \( \beta < \beta_1 \).

6. Results and Discussion

A theoretical investigation for the determination of
obliquely incident wave force at shallow water is
done on vertically submerged circular thin plate
suspected at three different configurations as
depicted in Fig. 1. For each type of configurations,
results are estimated up to six decimal places in order
to achieve accuracy. Besides these, investigation is
also done to find the effect of moment and force on
the plate at different incident angles (\( \theta \)) and at
different beach angles (\( \beta \)). Estimates are done at
different depth of water \( d=1, 2, 3, 4 \) & \( 5m \). But in this
presentation, results for each of the three types of
configurations are discussed considering \( d=1m, \) & \( 5m \)
only, since pattern of results are similar for \( d=2, 3 \) & \( 4 \).

Though numerical estimates for non-dimensional
horizontal force \( (F_x) \) & moment \( (M) \) for the three
types of vertically submerged circular thin plate have
been calculated for quite a number of incident angles,
only representative sets are shown in Tables 1-3,
taking \( \theta =5^\circ \) & \( 20^\circ \) and \( T=2 \) & \( 5s \). It is observed that

| Table 1—Values of non-dimensional horizontal force \( (F_x) \) for \( t=0.39s, \) \( d=1m, \) \( l=1mm, \) \( D=0.5m, \) \( x=1m, \) \( y=0.087m \) for \( \theta =5^\circ \) & \( x=1m, \) \( y=0.363m \) for \( \theta =20^\circ \) |
|---|---|---|---|---|---|
| \( H/d \) | \( \theta =5^\circ \) | \( \theta =20^\circ \) | \( \theta =5^\circ \) | \( \theta =20^\circ \) |
| Type I | | | | |
| 0.10 | 0.0693 | 0.0578 | 0.0225 | 0.0183 |
| 0.56 | 0.4044 | 0.1450 | 0.2766 | 0.1678 |
| 0.80 | 0.5896 | 0.0739 | 0.5074 | 0.2885 |
| Type II | | | | |
| 0.10 | 0.0572 | 0.0488 | 0.0218 | 0.0178 |
| 0.56 | 0.3312 | 0.1520 | 0.2652 | 0.1616 |
| 0.80 | 0.4813 | 0.1268 | 0.4856 | 0.2773 |
| Type III | | | | |
| 0.10 | 0.0503 | 0.0435 | 0.0214 | 0.0174 |
| 0.56 | 0.2903 | 0.1501 | 0.2585 | 0.1580 |
| 0.80 | 0.4210 | 0.1447 | 0.4728 | 0.2707 |

| Table 2—Values of moment \( (M) \) for \( t=0.39s, \) \( d=1m, \) \( l=1mm, \) \( D=0.5m, \) \( x=1m, \) \( y=0.087m \) for \( \theta =5^\circ \) & \( x=1m, \) \( y=0.363m \) for \( \theta =20^\circ \) |
|---|---|---|---|---|
| \( H/m \) | \( \theta =5^\circ \) | \( \theta =20^\circ \) | \( \theta =5^\circ \) | \( \theta =20^\circ \) |
| Type I | | | | |
| 0.10 | 0.0219 | 0.0182 | 0.0071 | 0.0058 |
| 0.56 | 0.1277 | 0.0458 | 0.0873 | 0.0530 |
| 0.80 | 0.1862 | 0.0233 | 0.1602 | 0.0911 |
| Type II | | | | |
| 0.10 | 0.0090 | 0.0077 | 0.0034 | 0.0028 |
| 0.56 | 0.0523 | 0.0240 | 0.0419 | 0.0255 |
| 0.80 | 0.0760 | 0.0200 | 0.0767 | 0.0438 |
| Type III | | | | |
| 0.10 | 0.0053 | 0.0046 | 0.0022 | 0.0018 |
| 0.56 | 0.0306 | 0.0158 | 0.0272 | 0.0166 |
| 0.80 | 0.0443 | 0.0152 | 0.0498 | 0.0285 |
$F_r$ & $M$ are very high for $\theta=5^\circ$ & $d=5m$, whereas $F_r$ initially increases and then decreases for $T=2s$, $\theta=20^\circ$ & $d=1m$. The values of $F_r$ are approximately same for $H/d=0.1$ in each case. Computed result of $M$ for $d=5m$, $\theta=5^\circ$ and $\beta=57.877^\circ$ is very high for type I compared to type II & III for $T=2s$ than $d=1m$ and $\beta=16.689^\circ$. It is also observed that $F_r$ & $M$ are very high for maximum beach slope when $T=2s$ & $\theta=5^\circ$.

Table 3—Values of non-dimensional horizontal force ($F_r$) for $t=0.39s$, $d=5m$, $l=1mm$, $D=0.5m$, $x=1m$, $y=0.087m$ for $\theta = 5^\circ$ & $x=1m$, $y=0.363m$ for $\theta = 20^\circ$

<table>
<thead>
<tr>
<th>$H/d$</th>
<th>$T=2$, $\beta=57.877^\circ$</th>
<th>$T=5s$, $\beta=14.297^\circ$</th>
</tr>
</thead>
</table>
|       | $\theta = 5^\circ$ | $\theta = 20^\circ$ | $\theta = 5^\circ$ | $\theta = 20^\circ$
| Type I | $F_r$= | $F_r$= |
| 0.10  | 0.1265 | 0.1265 | 0.0000 | 0.0000 |
| 0.58  | 0.6811 | 0.6811 | 0.1960 | 0.1960 |
| 0.84  | 1.8796 | 1.8796 | 0.5384 | 0.4390 |
| Type II | $F_r$= | $F_r$= |
| 0.10  | 0.0003 | 0.0003 | 0.0045 | 0.0045 |
| 0.58  | 0.0075 | 0.0075 | 0.1094 | 0.0894 |
| 0.84  | 0.0207 | 0.0207 | 0.3001 | 0.2449 |
| Type III | $F_r$= | $F_r$= |
| 0.10  | 0.00000 | 0.00000 | 0.0035 | 0.0029 |
| 0.58  | 0.00021 | 0.00021 | 0.0850 | 0.0695 |
| 0.84  | 0.00057 | 0.00057 | 0.2330 | 0.1901 |

Graph 2—Non-dimensional horizontal force ($F_r$) at different beach angles versus non-dimensional wave height ($H$) for the case of a vertically submerged circular thin plate under three different configurations. Here, $t=0.39s$, $d=1m$, $l=1mm$, $D=0.5m$, $x=1$, $y=0.087$ for $\theta = 5^\circ$ & $x=1$, $y=0.363$ for $\theta = 20^\circ$.

Graph 3—Maximum overturning moment ($M$) on different beach angles versus wave height ($H$) for the case of a vertically submerged circular thin plate under three different configurations. Here, $t=0.39s$, $d=1m$, $l=1mm$, $D=0.5m$, $x=1$, $y=0.087$ for $\theta = 5^\circ$ & $x=1$, $y=0.363$ for $\theta = 20^\circ$ and then decreases after $H =0.46$, 0.58 & 0.68 m for type I, II & III respectively.

Graph 4 shows the trend of non-dimensional horizontal force ($F_r$) versus dimensionless wave height ($H/d$) at different beach angles for $t=0.39s$, $\theta=5^\circ$ & $20^\circ$, $d=5m$ and $T=2 & 5s$. It is seen that horizontal force of type I is significantly higher for $T=2s$ compared to type II & III for any ($H/d$). And the magnitude of $F_r$ at $\theta=20^\circ$ is lower than at $\theta=5^\circ$.  

In Fig. 5 non-dimensional horizontal force \( (F_x) \) is plotted against dimensionless time \( (t/T) \) for \( x=1, \ y=0.087, \ d=1m \ & \ \theta=5^\circ \). It is observed that from the figure forces are gradually increasing from negative to zero and become positive between the ranges \( 0 \leq t/T \leq 0.4455 \) following which these forces are decreasing from positive to negative between the ranges \( 0.4460 \leq t/T \leq 1.0000 \). At higher amplitude \( (H=0.8) \) increasing rate of wave force is very rapid compared to lower amplitude \( H=0.1 \) in the range \( 0 \leq t/T \leq 0.4455 \).

Analyzing these figures for \( d=1m \ & 5m \), indicate that a surface-piercing circular thin plate (type I) experiences more wave energy compared to submerged circular thin plate & bottom-standing circular thin plate (type II & III) for \( t=0.39s, \ \theta=5^\circ \ & 20^\circ \). It indicates that wave energy gradually increases with increase in amplitude and they become convergent with increase of wave period, except in case of \( d=1m, \ \theta=20^\circ \), where wave energy gradually increases and then decreases for \( T=2s \). Wave energy is nearly same at \( H/d \) for \( T=5s, \ d=1m, \ \theta=5^\circ \ & 20^\circ \) of all the three types.

7. Concluding remarks

The present work represents a theoretical design procedure using linear wave theory for computation of wave force and moment experienced by vertically submerged circular thin plate on which obliquely incident wave force act at shallow depth of water. Morison’s equation is used for the determination of wave force employing Laplace equation and boundary value problems in the finite domain.

The variations of horizontal force and moment, against wave height are obtained for different values of angle of incident wave, different values of depth of water, different values of wave period and the different beach angles at the shore. It is observed that force and moment of the three types are gradually converging with the increase of wave period except for the case \( T=2s, \ \theta=20^\circ \) and \( d=1m \). Also it is observed that at lower wave period, force & moment on type I configurations are maximum compared to type II & III at any amplitude. Gradients of force and moment of the type I are extremely high at smaller wave period.

Here the plate in consideration may act as a piston of a floating pump, which is a part of entire design concept of a reciprocating pump. The pump may be used to store energy of ocean wave in the form of potential energy. Similar type of analysis as presented may be carried out with circular plates of different dimensions for further research work.

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