Fracture analysis for a non-homogeneous weldment with a crack parallel to the interface

Yong-Dong Li\textsuperscript{a,b}, Hong-Cai Zhang\textsuperscript{a} & Kang Yong Lee\textsuperscript{b,*}

\textsuperscript{a}Department of Mechanical Engineering, Academy of Armored Force Engineering, No.21, Du-Jia-Kan, Chang-Xin-Dian, Beijing 100072, P.R. China
\textsuperscript{b}School of Mechanical Engineering, Yonsei University, Seoul 120-749, South Korea

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Fracture modeling and analysis are of significance in the design and evaluation of weldments. Based on a new four-zone model, the present work performs anti-plane fracture analysis on a non-homogeneous weldment consisting of four different kinds of zones: base metal, heat affected zone (HAZ), fusion zone and weld metal. The HAZs and fusion zones are modeled as functionally graded materials. A crack parallel to the interface is assumed to be in the HAZ or weld metal. The crack problem is solved by the methods of Fourier integral transform and dual integral equation. Parametric studies on the stress intensity factors yield two conclusions for the current fracture model of the four-zone weldment. (i) The optimal value of the mismatch factor is 1.0. The increase in mismatch factor will decrease the driving force of a HAZ crack, but it will enhance that of a weld-metal crack. In engineering, mismatch factors too larger or too smaller than 1.0 should be avoided. (ii) The effect of non-homogeneity parameter embodies that of the relative stiffness of the crack-tip region. Stiffer crack-tip region may induce larger driving force for the crack.

Keywords: Weldment, Mismatch factor, Heat affected zone, Fracture, Functionally graded materials

Welded structures are widely used in engineering. Cracking is one of the most common failures of these structures. Therefore, fracture researches have been an active field in the designs and applications of welded structures. In most existing literatures, a weldment is often modeled as a tri-zone structure consisting of base metal, heat affected zone (HAZ) and weld metal\textsuperscript{1,2}. Under special cases, it is even simplified as a bi-zone structure without HAZ\textsuperscript{3}. However, experimental tests indicated that it is generally not like this in practice. On one hand, a thin but important fusion zone might form between HAZ and weld metal due to the high temperature of welding\textsuperscript{4}. On the other hand, the fusion and inter-diffusion between weld metal and base metals might give rise to local redistributions of material properties. As a result, HAZs and fusion zones may have continuously varying microstructures and properties\textsuperscript{5}, i.e., they may become non-homogeneous (or, functionally graded) materials.

Due to the non-homogeneity of HAZs and fusion zones, the methods used in fracture analyses of functionally graded materials (FGMs) can also be applied to study the fracture problems of weldments. There are many papers published in the field of fracture mechanics of FGMs. Their modeling, methods and main results provide good references for crack problems of weldments. Wang et al.\textsuperscript{6,7} first developed a new piecewise linear multi-layered model for fracture analysis of FGMs with arbitrarily varying properties and demonstrated its validity by applying this new model to solve a variety of mechanics problems in functionally graded materials and structures\textsuperscript{7-11}. Later, Guo and Noda\textsuperscript{12} put forth a multi-layered piecewise exponential model to simulate the arbitrary variation of material properties in FGMs. El-Borgi et al.\textsuperscript{13} analyzed an insulated interface crack in a functionally graded coating-substrate system under both thermal and mechanical loading by the method of singular integral equation. Kolednik et al.\textsuperscript{14} presented a quantity called "material inhomogeneity term" to characterize the inhomogeneity of elastic modulus in linear elastic, non-hardening and hardening elastic-plastic bimaterials, and evaluated the effective crack driving force by finite element simulation. Chen\textsuperscript{15} used the method of singular integral equation and the element-free Galerkin method to study two cases of thermal fracture problems of a graded orthotropic coating, one with exponential non-homogeneity and the other with...
more complex material distribution, and verified the good agreement between the numerical results and the analytical solutions. Zhou and Wang\textsuperscript{16} employed Fourier integral transform and the Schmidt method to investigate the interface debonding problem for a functionally graded strip sandwiched between two homogeneous layers of finite thickness subjected to an uniform tension. Chen and Zhong\textsuperscript{17} analyzed the plane elasticity problem for a functionally graded interfacial zone containing a crack between two dissimilar homogeneous materials by utilizing the Fourier transform technique and the transfer matrix method. Ding and Li\textsuperscript{18} studied the periodic interface cracks in a functionally graded coating-substrate structure by finite Fourier transform and compared the cases of a single crack and periodic cracks. Chen and Chue\textsuperscript{19} addressed the anti-plane problem of two bonded functionally graded finite strips, each of which contains an internal crack normal to the interface. Kashtalyan and Menshykova\textsuperscript{20} investigated the bending responses of coated plates subjected to transverse loading with a functionally graded interlayer between the substrate and the top coating, and examined the dependence of stress and displacement fields in the plate on the stiffness gradient in the coating.

By considering the non-homogeneity of HAZs and fusion zones, the research team of the present work established\textsuperscript{21} a four-zone model for a non-homogeneous weldment and investigated its fracture behaviour by assuming a crack to be perpendicular to the interfaces. For practical weldments, typical cracks might also be parallel to the interfaces. The present work continues to perform fracture analysis on such a four-zone non-homogeneous weldment under the assumption that a HAZ or weld-metal crack is parallel to the interface. The analytical methods of Fourier integral transform and dual integral equation are used, and the effects of the mismatch factors and non-homogeneity parameters are discussed.

**Problem Formulation**

A non-homogeneous weldment\textsuperscript{21} consisting of four zones: base metal, HAZ, fusion zone and weld metal is shown in Fig. 1. Every kind of zone except the weld metal has two specific zones numbered by $kj$ with $k = 1, 2, 3$ referring to the base metal, HAZ and fusion zone, and $j = 1, 2$ referring to the upper and lower specific zones. The weld metal is numbered by 40. For convenience, the quantities of these zones have the corresponding numbers as subscripts. It is assumed that the shear moduli of the HAZs and fusion zones are exponential functions of coordinate $y$\textsuperscript{21, 22}:

$$
G_{21}(y) = G_{40}e^{\beta_{21}(y-h_{21})}/M_{41}, \quad G_{31}(y) = G_{40}e^{\beta_{31}(y+c_1)} \\
G_{22}(y) = G_{40}e^{\beta_{22}(y+c_2)}/M_{42}, \quad G_{32}(y) = G_{40}e^{\beta_{32}(y+c_2)}
$$

... (1)

where $M_{41} = G_{40}/G_{11}$ and $M_{42} = G_{40}/G_{12}$ are mismatch factors. $c_1, c_2$ and $c_4$ are given in the Appendix. $\beta_{ij}$ ($i = 2, 3; j = 1, 2$) are non-homogeneity parameters determined by

$$
\beta_{31} = -[(\ln M_{41} + \beta_{21}(h_{211} + h_{212}))/h_{11}] \\
\beta_{32} = (\ln M_{42} - \beta_{22}h_{22})/h_{12}
$$

... (2)

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**Fig. 1**—A non-homogeneous weldment with a HAZ crack parallel to the interfaces (a) Schematic diagram of the weldment and (b) Variation of shear modulus
A crack with half-length \( a \) is in the HAZ-21 and parallel to the interfaces. The rectangular coordinate system \( \chi \nu y \) is set up with its origin at the crack center and the axis \( \chi \) along the crack. For simplicity, only the anti-plane problem is addressed here. Because the problem is symmetrical about the axis \( y \), it is sufficient to study only the right-half weldment. The boundary and continuity conditions are

\[
\tau_{yj}(x, h_{211} + h_1) = \tau_{yj}(x, -c_5) = 0 \quad \ldots (3)
\]

\[
[w]_{yj} = [w]_{yj} = 0 \quad (k = 1, 2, 3; j = 1, 2) \quad \ldots (4)
\]

\[
[w](x, 0) = 0, \quad x \geq a \quad \tau_y(x, 0) = -\tau_0, \quad 0 < x < a \quad \ldots (5)
\]

where \( \tau_y \) and \( w \) are the anti-plane stress and displacement. \( c_5 \) is given in the Appendix. The square bracket denotes the discontinuity of a physical quantity.

The anti-plane constitutive equation is

\[
\tau_k = G(y)w_k \quad (k = x, y) \quad \ldots (6)
\]

where and hereafter the subscripts following a comma denote partial differentiations.

The governing equations for all the zones are

\[
\nabla^2 w_{ij} = 0; \quad \nabla^2 w_{40} = 0; \quad \nabla^2 w_{ij} + \beta_{ij} w_{ij,y} = 0 \quad (l = 2, 3; \ j = 1, 2) \quad \ldots (7)
\]

The dual integral equation and its solution

Applying Fourier cosine integral transform to Eq. (7) with respect to \( x \) gives

\[
\begin{align*}
w_{ij} &= \frac{2}{\pi} \int_0^\infty \left[ A_{ij}(s) e^{y} + B_{ij}(s) e^{-y} \right] \cos(sx)ds \\
w_{40} &= \frac{2}{\pi} \int_0^\infty \left[ A_{40}(s) e^{y} + B_{40}(s) e^{-y} \right] \cos(sx)ds \\
w_{3j} &= \frac{2}{\pi} \int_0^\infty \left[ A_{3j}(s) e^{-p_j \nu} + B_{3j}(s) e^{-q_j \nu} \right] \cos(sx)ds \\
w_{21j} &= \frac{2}{\pi} \int_0^\infty \left[ A_{21j}(s) e^{-p_j \nu} + B_{21j}(s) e^{-q_j \nu} \right] \cos(sx)ds \\
w_{22j} &= \frac{2}{\pi} \int_0^\infty \left[ A_{22j}(s) e^{-p_j \nu} + B_{22j}(s) e^{-q_j \nu} \right] \cos(sx)ds \\
( j = 1, 2) \quad \ldots (8)
\end{align*}
\]

where \( p_{ij} \) and \( q_{ij} \) \((k = 2, 3; \ j = 1, 2)\) are known functions given in the Appendix. Substituting Eq. (8) into Eq. (6) yields the anti-plane stresses, and then from Eqs (3) and (4), all the unknown coefficient functions can be expressed as functions of \( A_{11}(s) \) and the following results are further obtained

\[
\begin{align*}
w_{211}(x, y) &= \frac{2}{\pi} \int_0^\infty A_{11}(s) \left[ g_1 e^{-p_{11} \nu} + g_2 e^{-q_{11} \nu} \right] \cos(sx)ds \\
w_{212}(x, y) &= \frac{2}{\pi} \int_0^\infty A_{11}(s) \left( \frac{p_{21} g_1 + q_{21} g_2}{p_{21} g_1 + q_{21} g_4} \right) \left[ g_1 e^{-p_{11} \nu} + g_2 e^{-q_{11} \nu} \right] \cos(sx)ds \\
\tau_{y11}(x, y) &= -\frac{2G_{21}(y)}{\pi} \int_0^\infty A_{11}(s) \left[ p_{21} g_1 e^{-p_{11} \nu} + q_{21} g_2(s) e^{-q_{11} \nu} \right] \cos(sx)ds \\
\end{align*}
\]

\[
\ldots (9) \quad \ldots (10)
\]

where \( g_m(m = 1, 2, 3, 4) \) are known functions given in the Appendix.

Substituting Eqs (9) and (10) into Eq. (5) yields a dual integral equation,

\[
\begin{align*}
&\int_0^\infty A(s) \cos(sx)ds = 0, \quad x \geq a \\
&\int_0^\infty sf(s) A(s) \cos(sx)ds = \pi \tau_0 M_4 e^{\beta_{21}h_1} / G_{40}, \quad 0 < x < a
\end{align*}
\]

\[
\ldots (11)
\]

where

\[
\begin{align*}
&f(s) = (p_{21} g_1 + q_{21} g_2)/(sQ) \\
&Q \text{ is a known function given in the Appendix.}
\end{align*}
\]

According to the Copson method, \( A(s) \) could be expressed by an unknown auxiliary function \( \phi(\lambda) \) in the form \( ^{\text{23}} \),

\[
A(s) = \frac{\pi \tau_0 a^2 M_4 e^{\beta_{21}h_1} / G_{40}}{J_0(sa\lambda)} \int_0^1 \phi(\lambda) J_0(sa\lambda) \lambda d\lambda \quad \ldots (13)
\]

where \( J_0(sa\lambda) \) is the first kind of Bessel function of the 0th order.

Equation (11) can be transformed into a Fredholm integral equation by using Eq. (13)
\[ \varphi(\lambda) = \int_0^1 \varphi(\xi) N(\lambda, \xi) d\xi = 1 \] \quad \ldots (14)

where

\[ N(\lambda, \xi) = -\xi \int_0^\infty A f(\gamma/a) - 1 / J_0(\gamma \lambda) J_0(\gamma \xi) d\gamma \] \quad \ldots (15)

Finally, \( \varphi(\lambda) \) in Eq. (14) can be expressed as a Liouville-Neumann series\(^{24} \)

\[ \varphi(\lambda) = \sum_{j=1}^n \varphi_j(\lambda) \] \quad \ldots (16)

where

\[ \varphi_0(\lambda) = 1; \quad \varphi_1(\lambda) = \int_0^1 N(\lambda, \xi) d\xi; \]
\[ \varphi_2(\lambda) = \int_0^1 \int_0^1 N(\lambda, \xi_1) N(\xi_1, \xi_2) d\xi_1 d\xi_2; \]
\[ \varphi_j(\lambda) = \int_0^1 \cdots \int_0^1 N(\lambda, \xi_1) N(\xi_1, \xi_2) \cdots N(\xi_{j-1}, \xi_j) d\xi_1 \cdots d\xi_j \]

Integrating Eq. (13) by parts gives

\[ \hat{A}(s) = \frac{\pi r_0 a M_{41} e^{\beta_2 h_{21}}}{sg_{40}} \left[ \varphi(1) J_1(sa) - \int_0^1 \frac{\lambda}{\lambda} J_1(sa) \varphi(\lambda) d\lambda \right] \] \quad \ldots (17)

From Eqs (17), (12) and (10), the stress \( \tau_{y211}(x, y) \) can be expressed by

\[ \tau_{y211}(x, y) = -a r_0 e^{\beta_2 h_{21}} \varphi(1) \text{Re} \left( \int_0^\infty J_1(\xi a) e^{i \xi d\xi} \right) \] \quad \ldots (18)

where \( z = x + yi \) is the complex illustrated in Fig. 2 and Re() denotes the real part of a complex.

Integrating Eq. (18) yields its asymptotic solution in the form

\[ \tau_{y211}(x, y) = \frac{\varphi(1) r_0 \sqrt{2} a}{\sqrt{2 \pi r_1}} \cos(\theta/2) \] \quad \ldots (19)

It is indicated that \( \varphi(1) \) can be regarded as the non-dimensional stress intensity factor (SIF).

**Results and Discussion**

With \( h_{21} \) fixed and the crack located at the center of HAZ-21, i.e., \( h_{211} = h_{212} \), the variation of the SIF with the non-homogeneity parameter and the mismatch factor is illustrated in Fig. 3. In Fig. 3a, the weldment is undermatched (i.e., the mismatch factor is less than 1.0), but in Fig. 3b, it is overmatched (i.e., the mismatch factor is larger than 1.0). Computation indicates that the effects of \( \beta_{22} \) and \( M_{42} \) on the SIF of the HAZ-21 crack are relatively negligible. Hence,

![Fig. 2—The complex and polar coordinates in the vicinity of a crack](image-url)

![Fig. 3—Effects of \( \beta_{21} h_{21} \) and \( M_{41} \) on the SIF of a HAZ-21 crack (\( h_{11}/h_{21}=30; \ h_{21}/h_{21}=0.01; \ h_{21}/h_{21}=2; \ h_{22}/h_{22}=1; \ a/h_{21}=0.2; \ h_{211} = h_{212}; \ j=1, 2 \) (a) \( M_{41} < 1 \) (\( M_{42}=0.5; \ \beta_{21} h_{21} = -0.1 \)) and (b) \( M_{41} > 1 \) (\( M_{42}=2.1; \ \beta_{22} h_{22} = 0.1 \)))](image-url)
in Fig. 3, the values of $\beta_{22}$ and $M_{42}$ are fixed, and only the effects of $\beta_{21}$ and $M_{41}$ are illustrated.

Figure 3 shows that in both the undermatching and overmatching cases, the SIF decreases with the increasing non-homogeneity parameter $\beta_{21}$. When the mismatch factor is given, the relative stiffness of the crack-tip region decreases with the increasing non-homogeneity parameter $\beta_{21}$ (Refer to Fig. 2 (a) and (b) of Li et al.). Therefore, the higher the relative stiffness of the crack-tip region, the larger the SIF.

Figure 3 also indicates that the SIF decreases with the increasing mismatch factor $M_{41}$. The SIF of the overmatching case is notably less than that of the undermatching case. It can be concluded that overmatching is more advantageous than undermatching to reducing the driving force of the HAZ crack parallel to the interface.

Besides the HAZ, the weld metal is another zone prone to cracking. For comparison, the present work also investigates a weld-metal crack parallel to the interfaces. With $h_{21}$ fixed, $M_{41} = M_{42} = M$, $|\beta_{21}| = |\beta_{22}| = \beta$ and the crack located at the center of the weld metal, the effects of $M$ and $\beta$ on the SIF are shown in Fig. 4, where Fig. 4(a) and 4(b) correspond to the undermatching and overmatching cases, respectively.

It is indicated by Fig. 4 that in the undermatching case the SIF increases with the increasing non-homogeneity parameter $\beta$, but in the overmatching case, the SIF decreases with the increasing $\beta$. When the mismatch factor is given, the relative stiffness of the crack-tip region increases with the increasing $\beta$ in the undermatching case, however, it decreases with the increasing $\beta$ in the overmatching case (Refer to Fig. 2 (a) and (b) of Li et al.). Therefore, the effect of $\beta$ embodies that of the relative stiffness of the crack-tip region.

It is also revealed by Fig. 4 that the SIF increases with the increasing mismatch factor. The SIF in the overmatching case is remarkably larger than that in the undermatching case. From this, it can be concluded that undermatching is more advantageous than overmatching to reducing the driving force of a weld-metal crack parallel to the interface.

It follows from Figs 3 and 4 that to increase the mismatch factor is beneficial to reducing the driving force of a HAZ crack, but unfortunately it meanwhile enhances the driving force of a weld-metal crack. Hence, too large or too small values of the mismatch factors should be avoided in engineering design.

In order to find the optimal value of the mismatch factor, the effects of the mismatch factors on SIFs of a HAZ crack and a weld-metal crack are compared in Fig. 5, where the crack is assumed to be at the center.
of the HAZ or the weld metal, \( M_{41} = M_{42} = M \) and \( \beta_{21} = \beta_{22} = 0 \). It is indicated that the optimal value of the mismatch factor seems to be 1.0. Generally, a weldment with mismatch factor just being 1.0 is often unavailable in practice. Therefore, a weldment with mismatch factor close to 1.0 is preferable for engineering designs.

Conclusions
Anti-plane fracture analysis is performed for a non-homogeneous weldment by the methods of integral transform and dual integral equation. Parametric studies on the stress intensity factor yield two conclusions for the current fracture model of the four-zone weldment.

(i) To increase the mismatch factor will decrease the driving force of the HAZ crack, but unfortunately it will enhance the driving force of the weld-metal crack. The optimal value of the mismatch factor is found to be 1.0. In engineering design, we should avoid choosing mismatch factor too larger or too smaller than 1.0.

(ii) The effect of non-homogeneity parameter embodies that of the relative stiffness of the crack-tip region. With other conditions unchanged, if the relative stiffness of the crack-tip region is decreased by the variation of non-homogeneity parameters, the SIF will be reduced.

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References

Appendix:

\[
\begin{align*}
\mathcal{A} &= \mathcal{B} + \mathcal{C} + \mathcal{D} + \mathcal{E}
\end{align*}
\]