Seismic performance evaluation methodologies for civil engineering structures

C Antony Jeyasehar\textsuperscript{a}, K Sathish Kumar\textsuperscript{b*}, K Muthumani\textsuperscript{b} & N Lakshmanan\textsuperscript{b}

\textsuperscript{a}Department of Civil and Structural Engineering, Annamalai University, Annamalai Nagar 608 002, India
\textsuperscript{b}Structural Engineering Research Centre, Council of Scientific and Industrial Research, Taramani, Chennai 600 113, India

Received 2 June 2008; accepted 21 April 2009

There is a strong need for evaluating the adequacy of seismic performance of civil engineering structures following the damage and collapse of numerous structures during recent earthquakes. In addition, the adequacy of seismic performance of the older structures in regions of high seismicity, which are designed prior to the advent of revised seismic design codes, is also a matter of growing concern. Seismic performance evaluation of civil engineering structures is an iterative process involving alternate stages of experimentation and computation. Several experimental methodologies including quasi-static testing, effective force testing, shake table testing, pseudo dynamic testing and real time dynamic hybrid testing and computational methodologies based on different numerical time stepping procedures are used to simulate and evaluate the seismic performance of civil engineering structures. In this paper a broad overview on the existing computational and experimental seismic performance evaluation methodologies is reported. The paper also presents the genesis, development and mathematical formulation of the pseudo dynamic testing method in detail and highlights the merits and demerits of the method over the other experimental seismic performance evaluation methods.

Keywords: Seismic performance, performance evaluation, earthquake engineering, shake table testing, effective force testing, pseudo dynamic testing, real time dynamic hybrid testing.

Earthquakes are potentially devastating natural events which threaten lives, destroy property, and disrupt life-sustaining services and societal functions. India has a very real earthquake problem. Kashmir 2005, Sumatra 2004, Bhuj 2001, Chamoli 1999, Jabalpur 1997, Lattur 1993, Uttarkashi 1991, Bihar 1988 the past two decades have seen devastating earthquakes striking India with frightening regularity. India’s four recently upgraded seismic zones as per the Indian standard code of practice IS 1893:2002\textsuperscript{1} also emphasize that 59\% of the land area in India is under moderate to severe earthquake hazard that assumes criticality in the context of all scales of the built environment. There are approximately 12 crore buildings in seismic zones III, IV and V. Most of these buildings are not earthquake resistant and are potentially vulnerable to collapse in the event of a high intensity earthquake. Hence, there is a need for evaluating the adequacy of seismic performance of civil engineering structures in view of the damage and collapse of numerous structures during recent earthquakes. In addition, the adequacy of seismic performance of the older structures in regions of high seismicity, which were designed prior to the advent of revised seismic design codes\textsuperscript{1}, is also a matter of growing concern.

The simulation of seismic response of a civil engineering structure has become a routine in the design of modern construction. Most simulations are done using computational tools which were verified by alternative analysis techniques or by experiments. The inelastic response of structures is very difficult to assess. The time domain numerical simulation of structural performance under seismic loads is usually carried out by using either modal super position (for elastic structures), or by direct integration methods\textsuperscript{2} (for inelastic structures). Appropriate assumptions have to be made in order to predict and calculate the seismic performance of the simulated structure. In particular, the direct integration methods utilized in dynamic testing are actually performed step by step. Not only the analytical errors are accumulated gradually, but the selection of sampling periods also affects the accuracy and stability of the computational integration process. More modern techniques based on state space approach\textsuperscript{3,4} can be formulated using system transition matrices derived

\textsuperscript{1}For correspondence: (E-mail: ksk_sercm@yahoo.co.in / ksk@sercm.org)
from exact solutions. Such solutions are exact for elastic structures and present minimal errors for inelastic structures. Most recently analytical techniques based on Hamilton-Lagrangian formulations proved that inelastic problems with severe degradation, sudden breaks and repetitive impacts, as well as progressively collapsing structural assemblies can be solved with stable solutions using energy minimization techniques. These techniques and other computational formulations developed in recent years still need experimental verification and identification of unknown phenomena neglected in modeling.

**Behaviour of a structure during an earthquake**

During an earthquake, a finite quantity of energy is input into a structure. This input energy is transformed into both kinetic and potential energy, which must be either absorbed or dissipated through some mechanism. If there were no damping in the structure, vibrations would exist for all time. However, there is always some level of inherent damping within the structure, which withdraws energy from the system and therefore reduces the amplitude of vibration until the motion ceases. This inherent damping may not be always sufficient to protect the structure from large earthquakes. Under these circumstances the structure’s seismic performance can be improved if a portion of the input energy being absorbed or dissipated, not by the structure itself, but by some other type of energy dissipation mechanism. This is made clear by considering the conservation of energy relationship

\[ E = E_k + E_s + E_h + E_d \]  

Where, \( E \) is the absolute energy input from the earthquake motion, \( E_k \) is the absolute kinetic energy, \( E_s \) is the recoverable elastic strain energy, \( E_h \) is the irrecoverable energy dissipated by the structural system through inelastic deformations, and \( E_d \) is the energy dissipated by the supplemental damping devices. Since the first two energy components \( E_k \) and \( E_s \) are recoverable in nature, the seismic performance of the structure during an earthquake entirely depends on either \( E_h \) or \( E_d \) or both.

The absolute energy input \( E \) represents the work done by the total base shear force at the foundation on the ground displacement. It, thus, contains the effect of the inertial forces of the structure. In the conventional design approach, acceptable structural performance is accomplished by the occurrence of inelastic deformations at weak plastic hinge locations. This has a direct effect of increasing energy \( E_h \). It also has an indirect effect. The occurrence of inelastic deformation results in softening of the structural system which itself modifies the absolute input energy. In effect, the increased flexibility acts as a filter which reflects a portion of the earthquake energy. The significant result is that it leads to reduced acceleration and reduced strains in regions away from the plastic hinges.

**Seismic Performance Evaluation Methodologies**

Simulation of seismic performance of structures under earthquake loads is usually performed experimentally or computationally. Experimental results are often used to develop and calibrate computational models of structures and structural components. These computational models are used to predict the response of structures. Further experiments are then performed to validate and refine the computational models. Seismic performance evaluation is thus an iterative process involving alternate stages of experimentation and computation.

**Computational methodology and issues**

Computational solution of the equation of motion for a civil engineering structure is usually complex if the excitation/applied force or ground acceleration varies arbitrarily with time as in the case of a seismic loading or if the system is nonlinear. Such problems can only be tackled by numerical time stepping methods for integration of differential equation of motion for evaluating the seismic performance of the structure. A vast body of literature, including major chapters of several text books, exists about these methods for solving various types of differential equations that arise in the broad subject area of applied mechanics. The literature includes the mathematical development of these methods; their accuracy, convergence and stability properties; and computer implementations. Only a brief presentation of few methods that are especially useful in the seismic response evaluation of civil engineering structures is presented in the paper. These include: (i) method based on interpolation of the excitation function; (ii) method based on finite difference expressions of velocity and acceleration and (iii) method based on assumed variation of acceleration.

**Method based on interpolation of excitation**

This method is a highly efficient numerical procedure can be better used for linear systems by
interpolating the excitation/applied force over each
time interval and developing an exact solution for a
linear system\(^2\). If the time intervals are short, linear
interpolation is satisfactory. The seismic response
of the structure over the time interval is computed as the
sum of three parts namely, free vibration response due
to initial displacement and initial velocity, steady state
response to the step force with zero initial conditions,
and steady state response to ramp force with zero
initial conditions. This numerical procedure is
especially useful when the excitation/applied force is
defined at closely spaced time intervals as in the case
of an earthquake ground acceleration so that the linear
interpolation is perfect. The major drawback is that
the method is difficult for multi degree of freedom
systems unless their response is obtained as
superposition of modal responses. However, this
method can be conveniently adapted for single degree
of freedom systems.

Method based on finite difference (Central difference method)

This method is based on a finite difference
approximation\(^2\) of the time derivatives of dis-
placement namely, velocity and acceleration. The
central difference expressions for velocity \(v_i\) and
acceleration \(a_i\) at time \(t\) are arrived taking constant
time steps \(\Delta t = \Delta t\). Then the solution for response \(d_{i+1}\)
at time \(i+1\) is determined from the equation of motion
at time \(t\) without using the equation of motion at time
\(i-1\). The elastic and damping forces can be computed
explicitly using known displacements \(d_i\) and \(d_{i-1}\). Such
methods are called explicit methods. The main
drawback of this central difference method is that the
method predicts incorrect response due to numerical
round off when the time step chosen is too small
resulting in numerical instability. The specific
requirement for numerical stability in central
difference method is \((\Delta t/T_n)<(1/\pi)\) and for seismic
applications a time step typically \((\Delta t/T_n)<0.1\) predict
accurate responses.

Method based on assumed variation of acceleration (Newmark’s method)

This time stepping method developed by
Newmark\(^2\) is based on predictor equations for velocity
\(v_{i+1}\) and displacement \(d_{i+1}\) at time \(i+1\) in terms of
Newmark parameters \(\gamma\) and \(\beta\). The parameters \(\gamma\) and \(\beta\)
deﬁne the variation of acceleration over the time step
and determine the stability and accuracy characteristics. The variation in acceleration over the
time step is assumed to be linear for values \(\gamma = \frac{1}{2}\) and
\(\beta = 1/6\) and for values \(\gamma = \frac{3}{2}\) and \(\beta = 1/4\) average
acceleration is assumed. In general, typical selection
of \(\gamma = \frac{1}{2}\) and \(1/6 \leq \beta \leq 1/4\) is found satisfactory from all
points of view, including that of accuracy. The two
predictor equations combined with the equilibrium
equation at the end of the time step, provide the basis
for computing \(d_{i+1}\), \(v_{i+1}\) and \(a_{i+1}\) from the known \(d_i\), \(v_i\)
and \(a_i\). The major drawback of this method is that the
method is iterative due to its implicit nature. However, this drawback can be eliminated by
suitably modifying the Newmark’s original
formulation to make it explicit. The specific
requirement for numerical stability in Newmark
method is \((\Delta t/T_n)<0.551\) for linear acceleration
method. Whereas the average acceleration method is
stable for any value of \(\Delta t\), however, the method is
accurate only if \(\Delta t\) is small.

Experimental methodologies and issues

Several experimental methodologies are used to
simulate and evaluate the seismic performance of
structures and structural systems. These include
(i) quasi-static testing; (ii) shake table testing;
(iii) effective force testing; (iv) pseudo dynamic
testing; and (v) real time dynamic hybrid testing.

Quasi-static testing

In quasi-static testing method\(^7\), the test structure is
subjected to slowly changing prescribed forces or
deformations by means of hydraulic actuators. Inertial
forces within the structure are not considered in this
method. The purpose of this elementary test method is
to observe the material behaviour of the structural
elements, components, or joints when they are
subjected to cycles of loading and unloading during a
seismic event.

Shake table testing

A shake table, on the other hand, can realistically
simulate the effects of seismic forces on the test
structure. Shake table testing\(^7\) is one of the
experimental methods adopted for evaluating the
seismic performance of structures subjected to
simulated earthquake motions. This is towards
developing and validating new design and
construction methodologies with improved seismic
resistance and also for bench-marking new analytical
tools and software. Essentially in shake table testing,
the three basic dynamic forces namely inertial, elastic
and damping forces are induced in the tested
structure. Such a pure experimental seismic
performance evaluation of structures necessitates the
use of sophisticated and expensive dynamic actuators and control systems. Also, it is difficult to design large shake tables capable of reproducing actual ground motions, particularly when simulating multi axial earthquakes. Among the reasons limiting the simulation of realistic effects are the deformability and inertia of the shake table, its characteristic modes of vibration, the devices needed to carry the dead load of test specimen and overturning moments without impeding the table’s motion, the friction of the bearings, the physical capabilities of the hydraulic actuators, and to a lesser extent the limitations in the control devices.

**Effective force testing**

The effective force testing method\(^7\) is based on applying dynamic forces to a test structure that is anchored rigidly to an immobile ground; these forces are proportional to the prescribed ground acceleration and the local masses. The deflections measured in this test correspond to the motions of the structural points relative to the ground that would have been observed had the specimen been subjected to the actual earthquake at its base.

**Pseudo dynamic testing**

In the absence of an expensive shake table facility, it is possible to simulate the three force parameters using a static actuator through application of an equivalent pseudo dynamic force system by computation of inertial forces in the back-ground. For such a hybrid methodology, a specialized algorithm based on an appropriate mathematical model\(^8\) is needed for the off-line time integration and computation of inertial forces such that the forces are applied statically through static actuators. Restoring forces offered by the structure is experimentally evaluated on-line at each time step and reflects the actual in-elastic and energy dissipation characteristics of the structure. The pseudo dynamic testing method\(^9,11\) resembles the quasi-static method in that it also consists in applying slowly varying forces to the test structure. However, during testing, the motions and deformations observed in the test structure are used to infer the inertial forces that the structure would have been exposed to during the actual earthquake; this information is then fed back into a control engine so as to determine and adjust the effective dynamic forces that must be applied onto the structure\(^12\). These pseudo-dynamic forces are typically accomplished by means of actuators pushing against a large reaction wall. This alternate seismic performance evaluation methodology is picking up in the recent years and it is essential for to-days needs of growing India with enhanced seismic risk. This method has the advantage of testing large and tall test structures with center of mass well above the base which are normally cannot be tested on a shake table for evaluating their seismic performance. As this method adopts application of dynamic forces in an equivalent static mean through static actuators, close monitoring of the structural behaviour including crack initiation, crack growth and stiffness degradation is also becomes possible. The draw back in such a hybrid method is the lack of simulation of strain rate effects which may not be critical under seismic loads. Also the method is time consuming due to its iterative nature.

**Real time dynamic hybrid testing**

This structural simulation method involves the combined use of shake tables, actuators, and computational engines. The structure to be simulated is divided into one or more experimental and computational substructures. This substructure concept of testing was developed in the eighties and formulated by numerous researchers\(^10,12\). The interface forces between the experimental and computational substructure are imposed by actuators and resulting displacements and velocities are fed back to the computational engine. The earthquake ground motion can be applied to the experimental substructures by actuators as interpreted displacements (as in pseudo-dynamic technique) or by one or more shake tables. Uniqueness of this method is force based sub structuring. Since the shake tables induce inertia forces in the experimental substructures, the actuators have to be operated in dynamic force control as well. The resulting experimental-computational infrastructure is more versatile than the previously deployed techniques. The real time dynamic hybrid simulation\(^13,14\), a form of substructure testing techniques, allows only parts of the structure for which the analytical understanding is incomplete to be tested experimentally. In contrast to the other testing methods the real time dynamic hybrid method allows substructures to be tested under dynamic conditions so that they can be subjected to realistic load histories. This real time dynamic method allows the rate dependent effects to be captured accurately which are another important advantage over pseudo-dynamic test method wherein rate effects can not be captured\(^14\).
Moreover, when real time evaluation of the structure is combined with real time identification of properties the resulting computational system becomes a reliable tool for analytical studies.

**Pseudo Dynamic Testing Method**

**Introduction to pseudo dynamic testing method**

The pseudo dynamic test method is an experimental technique for evaluating the seismic performance of structural models in a laboratory by means of online computer control simulation. Using the right test equipment, it is a reliable, economic and efficient method for evaluation of large scale structures that are too large or too heavy to be tested by shaking table. The pseudo dynamic test method combines well-established structural dynamics analytical techniques with experimental testing. A test structure must be first idealized as a discrete-parameter system, so that the equations of motion for the system can be represented by second-order ordinary differential equations. Based on analytically prescribed inertial and viscous damping characteristics of the system, and on structural restoring forces directly measured during the test, the governing equation of motion for the test specimen can be solved by a step-by-step numerical integration method. The displacement response is computed, based on a specific earthquake excitation record, and is then imposed on the test structure by means of electro-hydraulic actuators. Thus, the quasi-statically imposed displacements of the test structure will resemble those that would have developed if the structure were tested dynamically. Figure 1 shows a typical view of pseudo dynamic testing arrangement on a building frame. Figure 2 shows a typical view of pseudo dynamic testing laboratory.

Pseudo dynamic testing combines features of quasi-static and shake table testing, and numerical time history analysis. A traditional static testing set-up is used with a specimen fixed to the testing floor and with the relevant mass degrees of freedom (DOF), controlled by hydraulic actuators. Similar to a shake table, a pseudo dynamic test subjects a structure to a specific excitation, for instance, a seismic ground acceleration record. However instead of exciting the base, the pseudo dynamic test moves the pertinent DOFs such that the time history of the relative displacements between the base and DOFs are comparable to those which would have occurred, had the structure undergone the true base excitation. Such a displacement time history can only be determined before hand for a linear system, where the properties of the system ($M$, $C$ and $K$) remain constant throughout the excitation. But the structures under extreme loading may exhibit significant non-linear stiffness due to damage incurred during the excitation. Whereas the numerical simulations rely on certain hysteretic rules to trace the changing stiffness, based on some evolving system parameters, such as displacements or inter-storey drifts, pseudo dynamic tests account for the non-linear effects directly through experimentally measured restoring forces.

A general structure under dynamic excitation may be discretized into degrees of freedom and represented by the matrix equation of motion,

$$Ma + Cv + Kd = p$$  \(\ldots (2)\)

A quasi-static testing set-up in the laboratory allows convenient measurement of the displacements and forces which are related to the stiffness matrix $K$ by
Proper measurement of $K$ however requires a series of static tests one for each DOF considered. The fundamental insight of pseudo dynamic testing is the substitution of the term $Kd$, in the above equation by $r$, the restoring force vector. All of the components of $r$ can be measured in a single step and such measurements will be the effects of non-linearity in $K$. Pseudo dynamic testing measures the effects of non-linearities in $K$ directly without any explicit determination of $K$, whereas, a non-linear numerical simulation proceeds by explicit assembly of $K$ according to certain pre-defined rules. The remainder of the pseudo dynamic testing method follows directly from numerical integration techniques. A specific excitation record defines the load vector, $p$. The mass and viscous damping matrices are specified numerically to represent the prototype structure, but do not need to physically model in the laboratory. Displacement is related to its first and second time derivatives namely velocities and accelerations, using any of the various numerical integration techniques available.

**Genesis and developments in pseudo dynamic testing**

Pseudo dynamic test method was first proposed and used by Japanese researchers in 1975. A US-Japan cooperative earthquake research program in 1980s provided impetus for further development of the method with significant research efforts in the US occurring primarily at University of California, Berkeley and University of Michigan, Ann Arbor. At that time much of the research is focused on controlling higher mode error added to the numerical process can develop rapidly into spurious higher mode response.

The desire to test full scale structures with many DOFs and stiff, shear wall type structures led to much research effort towards controlling higher mode error propagation and introducing implicit integration to overcome the stability limit. Earlier method developed included, the introduction of numerical damping, and a novel, hybrid algorithm implemented as a combination of numerical and analog electronic methods. The most recent developments have focused on pseudo dynamic-specific adaptation of implicit numerical integration algorithms. Two such schemes have emerged as the most widely accepted. They are (i) $\alpha$-method based on Hilber- $\alpha$ integration and (ii) operator splitting (OS) algorithm.

The $\alpha$-method of Shing possesses extremely favorable error accumulation characteristics over a wide range of frequencies and prevents cyclic loading during iteration through a relaxation parameter. The OS algorithm requires no iteration by using an estimated tangent stiffness matrix and by introducing numerically a residual force imbalance correction. An excellent summary of this second generation of pseudo dynamic test methods given in references.
present a comparative results from pseudo dynamic tests on a three storied steel frame using both implicit method and central difference method. As a result of these developments, the use of pseudo dynamic testing for stiff, multi-storey specimens has become more common in both United States and Europe. A major pseudo dynamic experimental program at the University of California, San Diego has been conducted on a five storey, full scale reinforced masonry specimen. Two significant innovations developed during the course of this research include, ‘soft coupling’, to improve actuator control and the ‘Generated sequential displacement’ method to generalize the pseudo dynamic test beyond a single ground motion. Results of this program have been published in numerous forums. In Europe, Donea et al. reported on the testing of a four storey full scale reinforced concrete frame and a reduced scale series of bridge piers using sub-structuring. In Cornell University pseudo dynamic testing has been performed on a two storey two bay in-filled steel frame. All of the aforementioned tests used some form of the implicit integration schemes. However, recent testing by Negro et al. used the central difference algorithms on a four storey full scale reinforced concrete specimen.

Mathematical formulations

The pseudo dynamic test proceeds through a sequential numerical integration of the equation of motion, application of the specified displacements and measurement of the restoring forces. The numerical integration may be based on any of the Newmark-β family of algorithms or several adaptations developed specifically for pseudo dynamic testing. Buonopane proposes the following formulation with the explicit Newmark (β=0; γ = 0.5) algorithm. Fundamental displacement and velocity relationships are,

\[ d_{i+1} = d_i + (\Delta t)v_i + 0.5(\Delta t)^2 a_i \]  \hspace{1cm} \ldots (5)

\[ v_{i+1} = v_i + (\Delta t)(1 - \gamma)a_i + (\Delta t)\gamma a_{i+1} \]  \hspace{1cm} \ldots (6)

Substituting these into equation of motion, the acceleration \( a_{i+1} \), may be found by the equivalent system.

\[ a_{i+1} = (M^*)^{-1}.p^* \]  \hspace{1cm} \ldots (7)

Where the effective mass \( M^* \) and effective force \( p^* \) are given by,

\[ M^* = M + (\Delta t)\gamma C \]  \hspace{1cm} \ldots (8)

\[ p^* = p_i + r_{i+1} - C(v_i + \Delta t(1 - \gamma)a_i) \]  \hspace{1cm} \ldots (9)

To step forward in time domain, the target displacement \( d_{i+1} \) calculated from the previous equations, is applied to the structure and the restoring force vector \( r_{i+1} \) is measured. The effective force vector is calculated and used to determine the new acceleration. Finally, the new velocity is calculated from Eq. (5) and the process is repeated. Figure 3 shows the flow diagram of the pseudo dynamic testing scheme.

Advantages and disadvantages of pseudo dynamic testing

Review of various developments in pseudo dynamic testing reveals certain unique advantages and disadvantages of pseudo dynamic testing as compared to quasi-static or shake table testing and non-linear numerical analysis.

Advantages

(i) For multi-DOF systems, no assumptions on the distribution of seismic forces among the DOFs need to be made.

(ii) Full scale and large specimens may be tested with equipment requirement not much different than necessary for quasi-static testing.

![Fig.3— Flow diagram of a typical pseudo dynamic testing scheme](image-url)
(iii) Controlled testing speed allows for data acquisition from extensive instrumentation with modest electronics and careful recording of important information such as crack trajectories.

(iv) Specimen mass need not be accurately reproduced in the laboratory as it is modeled numerically.

(v) Effects of damage on behaviour are physically modeled with no numerical assumptions regarding degradation necessary.

(vi) Unique sub-structure tests are possible, where part of the prototype is built and tested in the laboratory, while the remainder is modeled numerically within the time integration loop.

**Disadvantages**

(i) Error propagation characteristics of numerical integration require excellent hydraulic control of actuators and tight tolerances on experimental error in displacement and force feedback.

(ii) Non-linear behaviour sensitive to strain rate cannot be reproduced without real-time pseudo dynamic testing.

(iii) Controlled testing speed and small integration time steps for numerical accuracy may cause excessive testing times.

(iv) Test response is often specific to a particular input motion.

Of course, many of these advantages and disadvantages applied to one or more of the other testing methods. Figures 4 and 5 respectively show the pseudo dynamic test facility and the real time pseudo dynamic test facility developed at National Centre for Research in Earthquake Engineering (NCREE), Taiwan. Similarly, Fig. 6 shows the pseudo dynamic test facility developed at Structural Engineering Research Centre (SERC), Chennai.

**Conclusions**

A broad overview of various experimental seismic performance evaluation methodologies for civil engineering structures including quasi-static testing, effective force testing, shake table testing, pseudo dynamic testing and real time dynamic hybrid testing is outlined in the paper. However, a detailed experimental investigation is needed to evaluate the relative reliability of the various parameters involved by conducting tests on a structure using different seismic performance evaluation methodologies explained. The paper also presents the genesis, development and mathematical formulation of the pseudo dynamic testing method in detail and also
highlights the merits and demerits of the method over the other experimental seismic performance evaluation methods.

**Nomenclature**

- $a_i$ = acceleration vector
- $a_{i+1}$ = acceleration at $i^{th}$ time step
- $a_{i+1}$ = acceleration at $i+1^{th}$ time step
- $a_{i-1}$ = acceleration at $i-1^{th}$ time step
- $C$ = damping matrix
- $d$ = displacement vector
- $d_i$ = displacement at $i^{th}$ time step
- $d_{i+1}$ = displacement at $i+1^{th}$ time step
- $d_{i-1}$ = displacement at $i-1^{th}$ time step
- $E$ = absolute energy input from the earthquake motion to the structure
- $E_k$ = absolute kinetic energy component of the structure
- $E_r$ = recoverable elastic energy component of the structure
- $E_u$ = irrecoverable inelastic energy component dissipated by the structure
- $E_d$ = energy component dissipated by supplemental damping devices
- $K$ = stiffness matrix
- $M$ = mass matrix
- $M'$ = effective mass matrix
- $p$ = external force vector
- $p_i$ = force at $i^{th}$ time step
- $p_{i+1}$ = force at $i+1^{th}$ time step
- $p_r$ = effective force
- $r$ = restoring force vector
- $T_n = n^{th}$ natural period of the structure
- $\Delta t$ = time step for the numerical integration process
- $V$ = velocity vector
- $v_i$ = velocity at $i^{th}$ time step
- $v_{i+1}$ = velocity at $i+1^{th}$ time step
- $v_{i-1}$ = velocity at $i-1^{th}$ time step
- $y, \beta$ = Newmark constants for the time integration process
- $\omega_n = n^{th}$ circular natural frequency of the structure

**References**