Fractional quantum Hall effect in graphene

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The electrons in most of the conductors can be described by non-relativistic quantum mechanics but the electrons in graphene behave as massless relativistic particles, called Dirac fermions, though their speed is given by the Fermi velocity. The relativistic nature of the energy dispersion relation of electrons in graphene modifies the inter electron interactions. This results in the observation of a number of peculiar properties e.g. anomalous quantum Hall effect. We study the fractional quantum Hall effect (FQHE) in graphene. The quantized Hall conductivity in graphene in FQHE is shown to be:

\[
\sigma_{\gamma\gamma} = \pm \frac{2n + 1}{2m(2n + 1) + 1} \frac{2e^2}{h}, \quad \text{where } n = 0, 1, 2, 3 \ldots
\]

This fascinating result shows that the FQHE in graphene has a sequence of states which is different from the sequence found in the 2D electron gas.

Keywords: Fractional quantum Hall effect, Graphene, Landau levels, Half effect

1 Introduction

Graphene\(^1,2\) is the recently discovered two-dimensional allotropic form of carbon. Its structure consists of a carbon honeycomb lattice made out of hexagons. The hexagons can be thought of benzene rings from which the hydrogen atoms were extracted. It is the building block for many forms of carbon allotropes e.g. three-dimensional (diamond, graphite), one-dimensional (nanotubes) and zero-dimensional\(^3\) (fullerenes). Graphite is obtained by the stacking of graphene layers. Diamond can be obtained from graphene under extreme pressure and temperatures by transforming the 2-dimensional sp\(^2\) bonds into 3-dimensional sp\(^3\) bonds. Carbon nanotubes are synthesized by graphene wrapping. Fullerenes can also be obtained from graphene by modifying the hexagons into pentagons and heptagons in a systematic way.

In graphene, electrons do not flow through the material as in silicon circuits, but on the surface\(^4\). Electrons moving around carbon atoms interact with graphene’s periodic potential which gives rise to new quasi-particles that have lost their rest mass (called massless Dirac fermions). These quasi-particles can be treated as electrons that have lost their rest mass or as neutrinos that acquired the electronic charge. Such massless charged particles have not been observed before. It is found that in graphene electrons travel much faster than electrons in other semiconductors. Hence, graphene can be used in place of silicon for making ultra-fast and stable transistors based on quantum physics. Further, it is found that graphene has a minimum electrical conductivity of the order of the quantum unit \(e^2/h\), even when the concentration of charge carriers is zero\(^5,6\). This is a peculiar property of graphene because in all other systems, the conductivity is zero if no charge carriers are present.

Graphene also shows very interesting behaviour in the presence of a magnetic field. Graphene shows an anomalous quantum Hall effect with the sequence shifted by 1/2 with respect to the standard sequence. It is usually observed at very low temperatures, typically below \(-243^\circ\text{C}\). The basic fact about the quantum Hall effect (QHE) is that the diagonal electric conductivity of two-dimensional electron system in a strong magnetic field is vanishingly small \(\sigma_{xx} \to 0\), while the non-diagonal conductivity is quantized\(^7\) in multiples of \(e^2/h\): \(\sigma_{xy} = ne^2/h\), when \(n\) is an integer (the integral quantum Hall effect, IQHE) and when \(n\) is a fractional number (the fractional quantum Hall effect\(^8,9\), FQHE). In recent experiments\(^5,6\), the integral quantum Hall effect is
observed in graphene. But in this case, it is found as \( \sigma_{xy} = \pm 4 \left( n + \frac{1}{2} \right) e^2 / h \). That is why; it is characterized as half-integer quantum Hall effect. This anomalous QHE is the direct evidence for Dirac fermions in graphene. The Quantum Hall effect in graphene has also been studied. Further, it is found that the QHE at room temperature can be observed in graphene. The superconductivity can be induced in graphene and the quantum Hall effect in it on the basis of supersymmetric (SUSY) quantum mechanics has been studied.

The recent observation of IQHE in graphene has lead to several questions: Does the fractional quantum Hall effect occur in graphene, and if so, what is its character? Does it resemble with the IQHE in graphene? Although the FQHE has not yet been observed in graphene, it has been explored theoretically in a number of papers. Possible fractional quantum Hall states in graphene have been discussed there. The possible supersymmetric structure of fractional quantum Hall effect in graphene is discussed by the authors which may be checked experimentally in future. The problem of electron-electron interactions in the presence of a large magnetic field in a honeycomb lattice is a complex problem that deserves a rigorous study. In this paper, we have studied the fractional quantum Hall effect in graphene. The quantized Hall conductivity in graphene in FQHE is shown to be:

\[
\sigma_{xy} = \pm \frac{2n + 1}{2m(2n + 1) + 1} \frac{2e^2}{h}, \quad \text{where} \quad n = 0, 1, 2, 3 \ldots
\]

2 Energy Spectrum of Graphene in Magnetic Field

Graphene has two atoms per unit cell. The low-energy band structure of graphene is described by cones located at two inequivalent Brillouin zone corners called the \( K \) and \( K' \) points. These points are defined as:

\[
\pm K = \left( \pm 4\pi / \sqrt{3a}, \ 0 \right) \quad \ldots (1)
\]

where \( a \) is the lattice constant, + sign is for the point \( K \) and – sign is for the point \( K' \). Near the points \( K \) and \( K' \), the electrons have a linear Dirac-Weyl (relativistic) type dispersion relation. The point \( K \) can be transformed into the point \( K' \) under the mirror reflection.

Let us apply the magnetic field to a graphene sheet taken in the xy plane, we assume that the magnetic field is homogeneous and given by:

\[
\vec{B} = \vec{\nabla} \times \vec{A}, \quad \ldots (2)
\]

where \( \vec{A} \) is the magnetic vector potential. In the presence of magnetic field, the Hamiltonian matrix can be written as:

\[
H = v_F \begin{pmatrix}
0 & P_x - iP_y \\
P_x + iP_y & 0
\end{pmatrix} \quad \ldots (3)
\]

where the covariant momentum is

\[
P_i = -i\hbar \partial_i + eA_i. \quad \ldots (4)
\]

In the zero field, the Hamiltonian given in Eq. (3) has a linear dispersion for both \( K \) and \( K' \) points. Let \( z = x - iy \) and symmetric gauge \( \vec{A} = (-B_y / 2, \ B_x / 2) \). In magnetic field, electrons undergo cyclotron motion and fill Landau levels (LL) successively. The Landau level raising and lowering operators can be written as:

\[
a^+ = \frac{l_B (P_x - iP_y)}{\sqrt{2\hbar}} \quad \ldots (5)
\]

and

\[
a = \frac{l_B (P_x + iP_y)}{\sqrt{2\hbar}}. \quad \ldots (6)
\]

where \( l_B \) is the magnetic length and the commutation relation \( [a, a^+] = 1 \) gives \( [P_x, P_y] = i\hbar^2 / l_B^2 \).

Now, the Hamiltonian becomes:

\[
H = \frac{\sqrt{2}}{l_B} \hbar v_F \begin{pmatrix}
0 & a \\
a^+ & 0
\end{pmatrix}, \quad \ldots (7)
\]

and

\[
H^2 = \frac{2\hbar^2 v_F^2}{l_B^2} \begin{pmatrix}
a^+a + 1 & 0 \\
0 & a^+a
\end{pmatrix}. \quad \ldots (8)
\]

Eigen vector of \( H^2 \) can be represented by

\[
\psi = \begin{pmatrix}
\alpha \eta_{n-1,m} \\
\beta \eta_{n,m}
\end{pmatrix}, \quad \text{where} \quad \eta_{n,m} \text{ are the standard LL eigen functions in graphene, } n, m = 0, 1, 2 \ldots \ldots \text{ is the LL index and } m \text{ is the angular momentum index. Each Landau level is fourfold degenerate. The corresponding wave functions for an electron can be written as:}
\]

\[
\psi(n \neq 0, m) = \frac{1}{\sqrt{2}} \begin{pmatrix}
-sgn(n)\iota \eta_{|n-1,m} \\
\eta_{|n,m}
\end{pmatrix}. \quad \ldots (9)
\]
and
\[ \psi(0,m) = \begin{pmatrix} 0 \\ \eta_{0,m} \end{pmatrix}. \] ...(10)

The corresponding Landau level energies:
\[ E_n = \pm \sqrt{\frac{2\hbar v_F e B |n|}{\epsilon}}, \] ...(11)

where the negative energy states corresponds to the Dirac hole states or the valence band states in the graphene. \[ ^{25} \]

3 Fractional Quantum Hall Effect in Graphene

When a piece of metal or semiconductor carrying a current is placed in a transverse magnetic field, an electric field is produced inside the material in a direction normal to both the current and the magnetic field. This phenomenon is known as the Hall effect. Hall performed his experiments at room temperature with moderate magnetic fields of less than one tesla (T). It is found that the Hall conductance varies linearly with \( B/\rho \), where \( \rho \) is the electron density and \( B \) is the magnetic field. This is the classical Hall effect. But, it is found that at low temperatures of only a few kelvin and high magnetic field (up to 30 T), the Hall conductance did not vary linearly with \( B/\rho \). Instead, it varied in a stepwise fashion:
\[ \sigma_{xy} = \frac{p}{2ps+1} \left( \frac{e^2}{h} \right) \equiv \nu \left( \frac{e^2}{h} \right), \] ...(12)

where
\[ p = 1, 2, ..., \quad s = 0, 1, 2, ....... \] ...(13)

Here \( \nu \) is the filling factor. It tells us how many energy levels are filled up. It is defined as the ratio between the electron density to the magnetic flux density.

When \( s = 0, \nu = p \) corresponds to the integral quantum Hall effect discovered by von Klitzing et al.\[^{30} \] When \( s > 0 \) corresponds to the FQHE discovered by Tsui et al.\[^{31} \] The case \( p = 1, \nu = 1/(2s+1) \) was explained by Laughlin.\[^{32} \] The general case \( s \geq 0, \ p \geq 1, \nu = \frac{p}{2ps+1} \) was explained by Jain using the idea of composite fermions (CF). The IQHE can be understood solely in terms of individual electrons in a magnetic field whereas the FQHE reflects new physics arising from the collective behaviour of all the electrons. The IQHE do not depend upon interactions between electrons whereas the FQHE depends upon the combined effects of the magnetic field and Coulomb interaction between electrons.

In graphene, the ground state of the system and the excitation spectrum are determined by the inter electron interactions. Now, we consider the Coulomb interactions. The zero energy state contains both electrons and holes. Due to the Coulomb attraction, electron-hole pairs make bound states and condense into the excitonic states which produce an excitonic gap. According to the SUSY spectrum, a single energy level contains up-spin and down-spin electrons belonging to different Landau levels. The form factor mixes these Landau levels\[^{17,34} \]. Coulomb interactions are different for electrons in different Landau levels. The effective Coulomb potential depends on spins in each energy level. Since gap energy is proportional to Coulomb energy or \( \sqrt{B} \), the FQHE appears for large magnetic field.

In high magnetic field, each electron captures an even number \( (2m) \) of quantized vertices to become a composite fermion (CF). The dynamics of composite fermions are described by an effective magnetic field given by:
\[ B' = B - 2\pi \left( 2m \right) \rho/\epsilon, \] ...(14)

where \( \rho \) is the electronic density in the lowest Landau level. The composite particles do not feel the external field \( B \) but the effective field \( B' \). Therefore, the FQHE of electrons can be considered as an IQHE of these composite particles.

For an electronic density \( \rho \), an effective filling factor \( f^* \) for the composite particles can be written as\[^{21} \]:
\[ f^* = \frac{2\pi \rho}{eB'}. \] ...(15)

In the lowest Landau level, the electron filling factor is:
\[ f = \frac{2\pi \rho}{eB}. \] ...(16)

Using Eq. (15) in Eq. (14), we can write:
\[ B' = B - 2m f^* B', \]
or \( B = B' \left( 1 + 2m f^* \right) \). ...(17)
From Eqs (15-17), we can write:
\[
f = \left( \frac{B}{B_f} \right) \left( \frac{1}{1 + 2mf^*} \right) = \frac{1}{1 + 2mf^*}
\]
or 
\[
f = \frac{f^*}{1 + 2mf^*}.
\] ...(18)

From the recent results\(^7,10,21\) about the IQHE in graphene, the quantized Hall conductivity is found to be:
\[
\sigma_{xy} = \pm 2(2n+1) \frac{e^2}{h} \equiv \pm 4 \left( n + \frac{1}{2} \right) \frac{e^2}{h}, \quad \ldots (19)
\]
where \(n\) is an integer. Here, the factor 2 is due to the spin degeneracy.

In the case of graphene, the effective filling factor \(f^*\) is associated with the integer quantum Hall effect of composite particles and if we ignore the spin then we can write:
\[
f^* = (2n+1), \quad n = 0, 1, 2, 3, \ldots \ldots \quad \ldots (20)
\]

Hence, the quantized Hall conductivity in graphene in FQHE can be written as:
\[
\sigma_{xy} = \pm \frac{2n+1}{2m(2n+1)+1} \frac{2e^2}{h}, \quad \ldots (21)
\]
where the negative energy states corresponds to the Dirac hole states or the valence band states in the graphene.

### 4 Results and Discussion

Now, consider Eq. (21) which represents the quantized Hall conductivity of graphene in FQHE.

(a) For \(m = 0\), Eq. (21) becomes Eq. (19) which represents the quantized Hall conductivity in IQHE in graphene.

(b) For \(n = 0\), Eq. (21) becomes:
\[
\sigma_{xy} = \pm \frac{1}{2m+1} \frac{2e^2}{h}, \quad \ldots (22)
\]

This equation explains the Laughlin sequence.

(c) But in this case Jain’s sequence is different from that of the 2D electron gas\(^33\).

The quantized Hall conductivity of fractional quantum Hall effect in graphene has been calculated. We have used the standard value of \(h/e^2 = 25813\ \Omega\). Hence \(e^2/h = 3.874 \times 10^{-5}\Omega^{-1}\). We use \(n \geq 0\) and \(m > 0\). The corresponding values of filling factors \((\nu)\) are found to be \(\pm 2/3, \pm 2/5, \pm 2/7, \pm 6/7, \pm 6/13, \pm 6/19, \pm 10/11, \pm 10/21, \pm 10/31\ldots\). Thus, the FQHE in graphene has a sequence of states which is different from the sequence found in the 2D electron gas.

Experimentally, different filling factors are achieved by varying the applied magnetic field at a fixed electron concentration. The values of Hall conductivity are presented in Table 1. The variation of \(\sigma_{xy}\) with respect to different values of filling factors is shown in Fig. 1. The unusual electronic dispersion of graphene is reflected in the variation of Hall conductance staircase. Due to the special lattice structure of graphene and Dirac nature of carriers, the

<table>
<thead>
<tr>
<th>(N)</th>
<th>(m)</th>
<th>(\nu = \pm \frac{2(2n+1)}{2m(2n+1)+1})</th>
<th>(\sigma_{xy} = \nu \left( \frac{e^2}{h} \right))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>(2/3 = 0.6666)</td>
<td>0.0000258</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>(2/5 = 0.40)</td>
<td>0.0000155</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>(2/7 = 0.2857)</td>
<td>0.0000110</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(6/7 = 0.8571)</td>
<td>0.0003332</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(6/13 = 0.4615)</td>
<td>0.000179</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>(6/19 = 0.3157)</td>
<td>0.000122</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(10/11 = 0.9090)</td>
<td>0.000352</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>(10/21 = 0.4761)</td>
<td>0.000185</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>(10/31 = 0.3225)</td>
<td>0.000125</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>(14/15 = 0.9333)</td>
<td>0.000362</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>(14/29 = 0.4827)</td>
<td>0.000187</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(14/43 = 0.3255)</td>
<td>0.000126</td>
</tr>
</tbody>
</table>

Fig. 1 — Quantized Hall conductivity of graphene in fractional quantum Hall effect for different filling factors.

\[\text{Table 1 — Calculation of filling factors and quantized Hall conductivity in graphene}\]
edge states play a key role in quantum Hall transport. The edge states of the \( v = 0 \) quantum Hall state carry no charge current but finite spin current in equilibrium\(^{35,36}\).

Our theoretical predictions may be checked experimentally in future. We think this fascinating result is due to the highly unusual nature of charge carriers in graphene. These facts lead to enrichment in the phenomenology of FQHE in graphene. The study of FQHE in graphene is a very challenging field both theoretically and experimentally in condensed matter physics as well as quantum field theory.

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References