Combined effects of Hall current and rotation on free convection
MHD flow in a porous channel

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A theoretical analysis of an oscillatory free convective MHD flow of a viscous incompressible and electrically conducting fluid in a vertical porous channel in the presence of Hall current has been carried out. The two insulating porous plates are subjected to a constant injection and suction. A uniform magnetic field is applied in the direction normal to the plates. The entire system rotates about the axis normal to the plates with uniform angular velocity $\Omega$. For small and large rotations, the dependence of the steady and unsteady resultant velocities and their phase differences on various parameters have been discussed in detail. The results show that both steady and unsteady resultant velocities increase rapidly from zero near the stationary plate and then approach to unity in the form of damped oscillations. For large values of rotations and injection/suction at the plates, a phase lag is also observed for both steady and unsteady phase angles.

Keywords: Oscillatory free convective flow, Rotating, Hall current, MHD flow

1 Introduction

The hydromagnetic convection with heat transfer in a rotating medium has been studied due to its importance in the design of magnetohydrodynamics (MHD) generators and accelerators in geophysics, the under ground water energy storage system, nuclear power reactors, soil sciences, astrophysics and MHD boundary layer control of reentry vehicles. In recent years, the problem of free convection has attracted the attention of a large number of researchers due to its diverse applications. On account of their varied importance, such a flow has been studied by Meyer\textsuperscript{1}. Attia and Kotb\textsuperscript{2} studied the MHD flow between two parallel porous plates. In the recent years, a number of studies have been done on the fluid phenomenon on earth involving rotation to a greater or lesser extent. Singh\textsuperscript{3} studied the unsteady free convective flow through rotating porous medium bounded by an infinite vertical porous plate. The exact solution of oscillatory Ekman boundary layer flow through a porous medium bounded by two horizontal flat plates in rotating system has been studied by Singh et al\textsuperscript{4}. The fluid flows in rotating channels have been studied by many researchers\textsuperscript{5-8}.

When the strength of the magnetic field is strong, one cannot neglect the effects of Hall currents. The Hall currents give rise to a cross flow making the flow three-dimensional. Free convective flows in the presence of Hall currents have been studied by many researchers\textsuperscript{9-14}.

In the present study, an oscillatory free convective MHD flow in a rotating vertical porous channel has been studied when the entire system rotates about an axis perpendicular to the planes of the plates and is also the axis along which a strong magnetic field of uniform strength is applied.

2 Formulation of Problem

Consider an oscillatory free convective flow of a viscous incompressible and electrically conducting fluid between two insulating infinite vertical porous plates distance $d$ apart. A constant injection velocity, $w_0$, is applied at the stationary plate $z^*=0$ and the same constant suction velocity, $w_0$, is applied at the plate $z^*=d$, which is oscillating in its own plane with a velocity $U^*(t^*)$ about a non-zero constant mean velocity $U_0$. The origin is assumed to be at the plate $z^*=0$ and the channel is oriented vertically upward along the $x^*$-axis. The channel rotates as a rigid body with angular velocity $\Omega^*$ about the $z^*$-axis perpendicular to the planes of the plates. A strong transverse magnetic field of uniform strength $H_0$ is also applied along the axis of rotation. The value of this uniform magnetic field is assumed to be unaltered
by making the necessary assumptions that guarantee the neglection of induced electric and magnetic fields. Using the relation \( \nabla \cdot H = 0 \) for the magnetic field \( H = (H_x, H_y, H_z) \), \( H_z = H_0 \) is obtained everywhere in the fluid (\( H_0 \) is a constant). If \( \mathbf{J} = (J_x^*, J_y^*, J_z^*) \) is the current density, from the relation \( \nabla \cdot \mathbf{J} = 0 \), one have \( J_z^* = \text{constant} \). Since the channel is non-conducting, \( J_z^* = 0 \) everywhere.

The generalized Ohm’s law, in the absence of the electric field (Mayer’s), is of the form:

\[
\mathbf{J} + \frac{\omega \tau_e}{H_0} (\mathbf{J} \times \mathbf{H}) = \sigma \left( \mathbf{\mu} \mathbf{V} \times \mathbf{H} + \frac{1}{e n_e} \nabla p_e \right)
\]

where \( \mathbf{V} \), \( \sigma \), \( \omega \), \( \tau_e \), \( e \), \( n_e \) and \( p_e \) are respectively the velocity, the electrical conductivity, the magnetic permeability, the cyclotron frequency, the electron charge, the number density of the electron and the electron pressure. Under the usual assumptions that the electron pressure (for a weakly ionized gas), the thermoelectric pressure and ion slip are negligible, one have from the Ohm’s law

\[
J_x^* + \omega \tau_e J_y^* = \sigma \mu_e H_0 v^*
\]

\[
J_y^* - \omega \tau_e J_x^* = -\sigma \mu_e H_0 u^*
\]

from which one obtains

\[
J_x^* = \frac{\sigma \mu_e H_0}{1 + m^2} (mv^* + v^*)
\]

\[
J_y^* = \frac{\sigma \mu_e H_0}{1 + m^2} (mv^* - u^*)
\]

where \( m = \omega \tau_e \) is the Hall parameter.

Since the plates are infinite in extent, all the physical quantities except the pressure depend only on \( z^* \) and \( t^* \). The physical configuration of the problem is shown in Fig. 1. Denoting the velocity components \( u^*, v^*, w^* \) in the \( x^*, y^*, z^* \) directions respectively and temperature by \( T^* \), the flow in the rotating system in the presence of Hall current is governed by the following equations:

\[
\frac{\partial w^*}{\partial z^*} = 0, \text{ which integrates to } w^* = w_0,
\]

\[
\frac{\partial u^*}{\partial t^*} + w_0 \frac{\partial u^*}{\partial t^*} = -\frac{1}{\rho} \frac{\partial p^*}{\partial x^*} + \nu \frac{\partial^2 u^*}{\partial z^*^2} + 2\Omega^* v^*
\]

\[
+ \frac{\sigma \mu_e^3 H_0^2 (mv^* - u^*)}{\rho(1 + m^2)} + g \beta (T^* - T_d^*)
\]

\[
\text{Fig. 1 — Physical configuration of the problem}
\]

where \( \nu \) is the kinematic viscosity, \( t^* \) is the time, \( \rho \) is the density and \( p^* \) is the modified pressure, \( \sigma \) is the electrical conductivity, \( T^* \) is the temperature, \( C_p \) is the specific heat at constant pressure, \( k \) is the thermal conductivity, \( g \) is the acceleration due to gravity, \( \beta \) is the coefficient of volume expansion and \( m \) is the Hall parameter.

The boundary conditions for the problem are

\[
\begin{align*}
\left\{ \begin{array}{l}
\eta \frac{z^*}{d} - t = \omega^* t^*, \\
\eta \frac{z^*}{d} - t = \omega^* t^* + \eta \left( T_0 - T_d \right) \cos \omega^* t^* \\
\end{array} \right. \\
\quad \text{at } z^* = 0,
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
u^* = 0,
\end{array} \right. \\
\quad \text{at } z^* = \frac{d}{2},
\end{align*}
\]

where \( \omega^* \) is the frequency of oscillation and \( \eta \) is a very small positive constant.

Now introduce the following non-dimensional quantities into Eqs (2)-(4):

\[
\eta = \frac{z^*}{d}, \quad \nu = \frac{\omega^* t^*}{U_0}, \quad \nu = \frac{\nu^*}{U_0}, \quad \Omega = \frac{\Omega^* d^2}{v} \quad (\text{rotation parameter}), \quad \omega = \frac{\omega^* d^2}{\nu} \quad (\text{frequency parameter}), \quad \lambda = \frac{\nu^* d^2}{\nu} \quad (\text{injection/suction parameter}), \quad M = H_0 d \sqrt{\frac{\sigma}{\mu}}
\]
(Hartmann number), \( Gr = \frac{\nu g \beta (T_0 - T_d)}{U_0 n_0^2} \) (Grashoff number), \( \theta = \frac{T^* - T_d}{T_0 - T_d} \) and \( Pr = \frac{\mu C_p}{k} \) (Prandtl number).

After combining Eqs (2) and (3) and by taking \( q = u + iv \), Eqs (2)-(4) reduce to:

\[
\frac{\partial q}{\partial t} + \lambda \frac{\partial q}{\partial \eta} = \frac{1}{1 + m^2} \left( q - U \right) - 2i\Omega (q - U) - M^2 (1 + im) (q - U) + Gr\lambda^2 \theta
\]

\[
\frac{\partial \theta}{\partial t} + \lambda \frac{\partial \theta}{\partial \eta} = \frac{1}{Pr} \frac{\partial^2 \theta}{\partial \eta^2}
\]

The boundary conditions (5) can also be written in complex notations as:

\[
\begin{align*}
q &= 0, \quad \theta = 1 + \frac{e^{i\theta}}{2} e^{i\eta} + q_{\text{e}}(\eta) e^{-i\eta} \quad \text{at} \quad \eta = 0 \\
q &= U(t) = 1 + \frac{e^{i\theta}}{2} e^{i\eta} + q_{\text{e}}(\eta) e^{-i\eta} \quad \text{at} \quad \eta = 1
\end{align*}
\]

In order to solve the system of Eqs (6) and (7) subject to the boundary conditions (8), it is assumed

\[
q(\eta, t) = q_0(\eta) + \frac{e^{i\theta}}{2} \left[ q_1(\eta) e^{i\theta} + q_2(\eta) e^{-i\theta} \right]
\]

\[
\theta(\eta, t) = \theta_0(\eta) + \frac{e^{i\theta}}{2} \left[ \theta_1(\eta) e^{i\theta} + \theta_2(\eta) e^{-i\theta} \right]
\]

Substituting Eqs (9) and (10) into Eqs (6) and (7) and comparing the harmonic and non-harmonic terms, one gets

\[
q_0^* - \lambda q_0' - S q_0 = -S - Gr\lambda^2 \theta_0
\]

\[
q_1^* - \lambda q_1' - (S + i\omega) q_1 = -(S + i\omega) - Gr\lambda^2 \theta_1
\]

\[
q_2^* - \lambda q_2' - (S - i\omega) q_2 = -(S - i\omega) - Gr\lambda^2 \theta_2
\]

\[
\theta_0^* - \lambda \text{Pr} \theta_0' = 0
\]

\[
\theta_1^* - \lambda \text{Pr} \theta_1' - i\omega \text{Pr} \theta_1 = 0
\]

\[
\theta_2^* - \lambda \text{Pr} \theta_2' + i\omega \text{Pr} \theta_2 = 0
\]

where \( S = \frac{M^2 (1 + im)}{1 + m^2} + 2i\Omega \) and \( \Lambda \) denotes the derivatives w.r.t. \( \eta \).

The corresponding transformed boundary conditions reduce to

\[
q_0 = q_1 = q_2 = 0, \quad \theta_0 = \theta_1 = \theta_2 = 1 \quad \text{at} \quad \eta = 0
\]

\[
q_0 = q_1 = q_2 = 1, \quad \theta_0 = \theta_1 = \theta_2 = 0 \quad \text{at} \quad \eta = 1
\]

The solutions of Eqs (11)-(16) under the boundary conditions (17) are

\[
q_0(\eta) = 1 - e^{\eta \eta} + A_1 (e^{\eta \eta} - 1) - A_2 (e^{\eta \eta} - e^{\lambda \eta}) + \frac{(e^{\eta \eta} - e^{\eta \eta})}{e^{\eta \eta} - e^{\eta \eta}} \left[ e^{\eta \eta} - A_1 (e^{\eta \eta} - 1) + A_2 (e^{\eta \eta} - e^{\lambda \eta}) \right]
\]

\[
q_1(\eta) = 1 - e^{\eta \eta} + B_1 (e^{\eta \eta} - e^{\eta \eta}) - B_2 (e^{\eta \eta} - e^{\eta \eta}) + \frac{(e^{\eta \eta} - e^{\eta \eta})}{e^{\eta \eta} - e^{\eta \eta}} \left[ e^{\eta \eta} - B_1 (e^{\eta \eta} - e^{\eta \eta}) + B_2 (e^{\eta \eta} - e^{\eta \eta}) \right]
\]

\[
q_2(\eta) = 1 - e^{\eta \eta} + C_1 (e^{\eta \eta} - e^{\eta \eta}) - C_2 (e^{\eta \eta} - e^{\eta \eta}) + \frac{(e^{\eta \eta} - e^{\eta \eta})}{e^{\eta \eta} - e^{\eta \eta}} \left[ e^{\eta \eta} - C_1 (e^{\eta \eta} - e^{\eta \eta}) + C_2 (e^{\eta \eta} - e^{\eta \eta}) \right]
\]

where

\[
\begin{align*}
m_1 &= \frac{\lambda \text{Pr} + \sqrt{\lambda^2 \text{Pr}^2 + 4i\omega \text{Pr}}}{2} \\
m_2 &= \frac{\lambda \text{Pr} - \sqrt{\lambda^2 \text{Pr}^2 + 4i\omega \text{Pr}}}{2} \\
m_3 &= \frac{\lambda \text{Pr} + \sqrt{\lambda^2 \text{Pr}^2 - 4i\omega \text{Pr}}}{2} \\
m_4 &= \frac{\lambda \text{Pr} - \sqrt{\lambda^2 \text{Pr}^2 - 4i\omega \text{Pr}}}{2}
\end{align*}
\]

\[
\begin{align*}
n_1 &= \frac{\lambda + \sqrt{\lambda^2 + 4S^2}}{2} \\
n_2 &= \frac{\lambda - \sqrt{\lambda^2 + 4S^2}}{2} \\
n_3 &= \frac{\lambda + \sqrt{\lambda^2 + 4(S + i\omega)}}{2} \\
n_4 &= \frac{\lambda - \sqrt{\lambda^2 + 4(S + i\omega)}}{2}
\end{align*}
\]
Results and Discussion

For the resultant velocities and the shear stresses of the steady and unsteady flow:

\[ u_0(\eta) + iv_0(\eta) = q_0(\eta) \quad \ldots (24) \]

and

\[ u_1(\eta) + iv_1(\eta) = q_1(\eta)e^{i\lambda} + q_2(\eta)e^{-i\lambda} \quad \ldots (25) \]

The Eq. (18) corresponds to the steady part which gives \( u_0 \) as the primary and \( v_0 \) as the secondary velocity components. The amplitude and the phase difference due to these primary and secondary velocities for the steady flow are given by

\[ R_0 = \sqrt{u_0^2 + v_0^2}, \quad \alpha_0 = \tan^{-1}(v_0/u_0) \quad \ldots (26) \]

The resultant velocity \( R_0 \) for the steady part is shown in Fig. 2 for small and large values of rotations of the vertical porous channel. The two values of the Prandtl number \( Pr \) as 0.71 and 7.0, are chosen to represent air and water respectively. It is clear from Fig. 2 that \( R_0 \) increases with the Grashoff number and suction velocity. It is interesting to note that the increase of the rotation of the channel and Hartmann number lead to an increase of \( R_0 \) near the stationary plate, but to a decrease near the oscillating plate. However, the effects of the Hall parameter and the Prandtl number are reversed, i.e. the amplitude \( R_0 \) decreases near the stationary plate and increases thereafter.

The phase difference \( \alpha_0 \) for the steady flow is shown graphically in Fig. 3 for small and large rotations. Fig. 3 shows that the phase angle \( \alpha_0 \) decreases with the increase of Grashoff number, Hartmann number, the suction/injection parameter, the rotation of the channel and Prandtl number. But the increase of Hall parameter leads to an increase of \( \alpha_0 \).

The amplitude and the phase difference of shear stresses at the stationary plate (\( \eta=0 \)) for the steady flow can be obtained as:

\[ \tau_{\eta \eta} = \sqrt{\tau_{\eta \eta}^2 + \tau_{\eta y}^2}, \quad \beta_0 = \tan^{-1}(\tau_{\eta y}/\tau_{\eta x}) \quad \ldots (27) \]
where

\[
\left( \frac{\partial \eta}{\partial \eta} \right)_{\eta=0} = \tau_0 + i\tau_{0y} = -n_2 + A_1 n_2 - A_2 (n_2 - \lambda \Pr) \\
+ \frac{(n_1 - n_2)}{e^{n_1} - e^{n_2}} \left(e^{n_1} - A_1 (e^{n_2} - 1) + A_2 (e^{n_2} - e^{k \Pr}) \right) 
\]

\[\ldots(28)\]

Here \(\tau_0\) and \(\tau_{0y}\) are, respectively, the shear stresses at the stationary plate due to the primary and secondary velocity components. The numerical values of the amplitude \(\tau_0\) of the steady shear stresses and the phase difference of the shear stresses at the stationary plate (\(\eta=0\)) for the steady flow are presented in Table 1.

The \(\tau_0\) increases and \(\beta_0\) decreases with the increase of Grashoff number, Hartmann number and suction/injection parameter. But the effects are reversed with the increase of Hall parameter, i.e. \(\tau_0\) decreases and \(\beta_0\) increases with the increase of \(m\). It is noted that both \(\tau_0\) and \(\beta_0\) increase due to an increase in the rotation of the channel. But an increase in the Prandtl number leads to decrease both in \(\tau_0\) and \(\beta_0\).

For the unsteady part, the resultant velocity or amplitude \(R_1\) is shown in Fig. 4 for rotation \(\Omega\) small and large. It is observed from Fig. 4 that \(R_1\) increases with the Grashoff number, Hall parameter and suction/injection parameter, but decreases with the increase of frequency of oscillations. It is interesting to note that increase in Hartmann number, rotation of the channel and the Prandtl number lead to an increase in \(R_1\) near the stationary plate, but to a decrease near the oscillatory plate.

The phase difference \(\alpha_1\) for the unsteady part is shown in Fig. 5. It is evident from Fig. 5 that increase in Grashoff number or Hartmann number or suction/injection parameter or rotation of the channel lead to a decrease in \(\alpha_1\) but \(\alpha_1\) increases with the increase of Hall parameter, Prandtl number and frequency of oscillations.

The resultant velocity or amplitude and the phase difference of the unsteady flow are given by:

\[R_1 = \sqrt{u_1^2 + v_1^2}, \quad \alpha_1 = \tan^{-1}(v_1 / u_1). \]

\[\ldots(31)\]

Table 1 — Values of \(\tau_{0x}\) and \(\beta_0\) for various \(Gr, M, m, \lambda, \Omega, \Pr\) and \(\Omega\)

<table>
<thead>
<tr>
<th>Gr</th>
<th>M</th>
<th>m</th>
<th>(\lambda)</th>
<th>(\Omega)</th>
<th>(\Pr)</th>
<th>(\tau_{0x})</th>
<th>(\beta_0)</th>
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<tbody>
<tr>
<td>5</td>
<td>2</td>
<td>1</td>
<td>0.5</td>
<td>10</td>
<td>0.71</td>
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<td>0.7279</td>
</tr>
<tr>
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<td>10</td>
<td>0.71</td>
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<td>0.6815</td>
</tr>
<tr>
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<td>1</td>
<td>0.5</td>
<td>10</td>
<td>0.71</td>
<td>5.2797</td>
<td>0.6395</td>
</tr>
<tr>
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</tr>
<tr>
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<tr>
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<td>0.71</td>
<td>12.558</td>
<td>0.7857</td>
</tr>
</tbody>
</table>

Fig. 4 — Resultant velocity \(R_1\) for small and large rotations due to \(u_1\) and \(v_1\) at \(t = (\pi/4)\).

Fig. 5 — Phase angle \(\alpha_1\) for small and large rotations due to \(u_1\) and \(v_1\) at \(t = (\pi/4)\).
Fig. 7 — Phase difference $\beta_1$ of unsteady shear stresses for small and large rotations at $t = (\pi/4)$

\[ \tau_{1r} = \sqrt{\tau_{1x}^2 + \tau_{1y}^2} \quad \beta_1 = \tan^{-1}(\tau_{1y}/\tau_{1x}). \quad \text{...(33)} \]

The amplitude $\tau_{1r}$ of the unsteady shear stress is shown graphically in Fig. 6 for small and large rotations. It is clear from Fig. 6 that $\tau_{1r}$ increases with the increase of Grashoff number, Hartmann number and the rotation of the channel. It is also observed that the increase of Prandtl number lead to an increase of $\tau_{1r}$ for small oscillations of frequency, but to a decrease for larger frequency of oscillations. However, the effects of the suction/injection parameter at the plates are reversed. It is interesting to note that with the increase of Hall parameter, $\tau_{1r}$ decreases for small oscillations of frequency, increases for moderate oscillations and thereafter decreases for larger frequency of oscillations.

The phase difference $\beta_1$ of the unsteady shear stress is shown graphically in Fig. 7 for small and large rotations. Fig. 7 clearly shows that $\beta_1$ decreases with the increase of Grashoff number, Hartmann number, the suction/injection parameter at the plates and the rotation of the channel, but increases with the increase of Prandtl number or Hall parameter.

4 Conclusions

The results obtained from the present study show that the resultant velocity of the steady part increases near the stationary plate and decreases near the oscillating plate as the rotation of the channel and
Hartmann number increase, however, the effects of the Hall parameter and the Prandtl number are reversed. The resultant velocity of the unsteady part increases with the Grashoff number, Hall parameter and suction/injection parameter, but decreases with the increase of frequency of oscillations. The phase angles of the steady and unsteady part of shear stresses decreases with the increase of Grashoff number, Hartmann number, the suction/injection parameter, the rotation of the channel. The amplitude of shear stresses of the unsteady part decreases for small oscillations of frequency, increases for moderate oscillations and thereafter decreases for larger frequency of oscillations with the increase of Hall parameter.

References