Gaussian minimum shift keying systems with additive white Gaussian noise

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In this paper, the basic properties of an I-Q Gaussian minimum shift keying (GMSK) with additive white Gaussian noise have been investigated. The probability of error for non-fading and fading channels of communication has been analyzed and discussed. The hardware realization is simple and straight forward like minimum shift keying (MSK) system. Some analytical results have been achieved and compared with machine computed GMSK results of others. The modular circuits are available for VLSI design. Thus, the system developed is suitable for VLSI design of GMSK.

Keywords: Gaussian minimum shift keying, Spectral density, Probability of error, Additive white Gaussian noise
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1 Introduction
Murota and Hirade\textsuperscript{1} proposed Gaussian minimum shift keying (GMSK) for mobile radio telephone services in 1981. Such modulation scheme is applicable to global system for mobile communications\textsuperscript{2} (GSM) and DCS 1800 due to its bandwidth efficiency and constant envelope. Later on, GMSK was analyzed and studied by many researchers\textsuperscript{3-17} I-Q GMSK modulator has been studied\textsuperscript{18} and in the present paper, some analytical results for spectral density of in-phase and quadrature-phase component of GMSK signal corrupted with noise have been achieved. The equation for probability of error for fading channel containing random attenuation factors has been derived. The hardware realization of GMSK modulator has been achieved through new binary data (+1, -1) achieved from the conventional binary data (0,1). Finally, the hardware realization of GMSK detector containing noise signal has been discussed.

2 Preliminaries
2.1 GMSK Modulation
Let the signal $s(t)$ is corrupted by Gaussian noise $n(t)$. The signal combined with noise, $s(t) + n(t)$ goes to the input of Gaussian filter. The output of Gaussian filter is given by $[s(t)+n(t)]^* h(t)$ where $h(t)$ is the Gaussian filter response and it is given by:

$$h(t) = \frac{A_0}{\sqrt{\pi}} e^{-[(t-t_0)/\beta]^2} \quad \ldots (1)$$

with $A_0$ is the amplitude and $t_0$ is group delay

$$\beta = \frac{2}{\ln(2)} \pi f_b(B_bT) \quad \ldots (2)$$

where $B_bT$ is design parameter and $f_b$ is the bit frequency.

where the signal $s(t)$ is a stream of rectangular pulses and it is given by:

$$s(t) = \sum b(n) p(t-nT_b) \quad \ldots (3)$$

where $p(t) = 1$ for $t \in (0,T_b)$

$= 0$ otherwise.

and $T_b$ is the symbol interval.

$$[s(t)+n(t)]^* h(t) = s(t)^* h(t) + n(t)^* h(t) = g(t) + n(t) \quad \ldots (4)$$

Now $g(t)$ may be expressed as:

$$g(t) = T \sum_{n=0}^{k} s(nT) h(t-nT) \quad \ldots (5)$$

i.e.

$$g(t) = T[b(0) p(t) h(t) + b(1) p(t-T) h(t-T) + b(2) p(t-2T) h(t-2T) + b(3) p(t-3T) h(t-3T) + \ldots] \quad \ldots (6)$$

The phase angle $\varphi(t)$ is given by:
\[ \varphi(t) = 2\pi f_m \left[ \int_0^t g(\tau) d\tau + \int_0^\tau n_0(\tau) d\tau \right] \]

where \( f_m \) is the modulated wave frequency and \( \omega_c \) is carrier wave angular frequency:

\[ f_m = \frac{1}{4T_b} = \frac{f_b}{4} \]

The desired modulated signal \( y(t) \) is given by:

\[ y(t) = \cos \left[ \omega_c(t) + \varphi_s(t) \right] + \cos \left[ \omega_c(t) + \varphi_n(t) \right] - \sin(\omega_c(t)) \sin(\varphi_n(t)) \]

\[ = \left[ I(t) + n_{oc}(t) \right] \cos(\omega_c t) - \left[ Q(t) + n_{os}(t) \right] \sin(\omega_c t) \]

where \( \varphi_s(t) \) and \( \varphi_n(t) \) are independent. Where

\[ I(t) = \text{Re} \left[ A_0 \exp \left( j2\pi f_m \int_0^t g(\tau) d\tau \right) \right] \]

\[ n_{oc}(t) = \text{Re} \left[ \exp \left( j2\pi f_m \int_0^t n_0(\tau) d\tau \right) \right] \]

\[ Q(t) = \text{Im} \left[ A_0 \exp \left( j2\pi f_m \int_0^t g(\tau) d\tau \right) \right] \]

\[ n_{os}(t) = \text{Im} \left[ \exp \left( j2\pi f_m \int_0^t n_0(\tau) d\tau \right) \right] \]

Substituting the value of \( g(\tau) \) from Eq.(6) and \( h(\tau) \) from Eq.(1) and integrating, we have:

\[ \int_0^t g(\tau) d\tau = \frac{b_A T}{2} \left[ \frac{p(T) \text{erf} \left[ \frac{\beta (T-t_0)}{T} \right]}{T} + p(0) \text{erf} \left( \frac{\beta t_0}{T} \right) \right] \]

\[ + b_1 \left[ p \left( T-t \right) \text{erf} \left[ \frac{\beta (T-t-\tau)}{T} \right] \right] \]

\[ + b_2 \left[ p \left( 2T-t \right) \text{erf} \left[ \frac{\beta (2T-t-\tau)}{T} \right] \right] + \ldots \]

\[ \ldots \text{(7)} \]

2.2 Spectral analysis

Power spectral density of signal \( I(t) \) is given by the relation:

\[ P(f) = \int_{-T/2}^{+T/2} \left\{ \cos \left( 2\pi f_m \right) \int_0^t g(\tau) d\tau \right\} e^{-j2\pi ft} d\tau \]

\[ \ldots \text{(8)} \]

Substituting the value of \( \int_0^t g(\tau) d\tau \) from Eq. (7) into Eq.(8) and integrating, we have:

\[ P(f) = \frac{1}{2} \left( 2\pi f + \pi f_m \beta A_0 b_0 T_b \right) \]

\[ \times \left[ \sin \left( 2\pi f T + \pi f_m \beta A_0 b_0 T \right) + p(0) \text{erf} \left( \frac{\beta t_0}{\sqrt{T}} \right) \right] \]

\[- \sin \left( \pi f_m \beta A_0 b_0 T \right) \left( p(0) t_0 \text{erf} \left( -\beta t_0 \right) \right) \]

\[ + \frac{e^{-\beta^2 t_0^2}}{\beta \sqrt{T}} + p(0) \text{erf} \left( \beta t_0 \right) \right] \]

\[ \ldots \text{(9)} \]

Similar expression can be derived for signal \( Q(t) \).

2.3 Error rate

The input power of signal \( I(t) \) is given by:

\[ S_i = \frac{1}{2} \cos^2 \left( \frac{\pi}{2T} \int_0^t g(\tau) d\tau \right) \]

\[ \ldots \text{(10)} \]

The input power of signal \( Q(t) \) is given by:

\[ S_i = \frac{1}{2} \sin^2 \left( \frac{\pi}{2T} \int_0^t g(\tau) d\tau \right) \]

\[ \ldots \text{(11)} \]

The output white noise spectral density:

\[ N_{0s} = N_{0c} = \left| H(f) \right|^2 \cdot \left( \frac{\eta}{2} \right) = A_0^2 e^{-2\alpha \eta} e^{-2j\frac{\pi}{\eta} \left( \frac{\eta}{2} \right)} \]

where \( \tau \) is a constant.

So, signal to noise ratio is given by:

\[ \frac{S_i}{N_{0c}} = \frac{\cos^2 \left( \frac{\pi}{2T} \int_0^t g(\tau) d\tau \right)}{A_0^2 e^{-2\alpha \eta} e^{-2j\frac{\pi}{\eta} \left( \frac{\eta}{2} \right)}} \]

\[ \ldots \text{(12)} \]

Probability of error for in-phase signal is given by:

\[ P_{ec} = \frac{1}{2} \text{erfc} \left( \frac{\cos^2 \left( \frac{\pi}{2T} \int_0^t g(\tau) d\tau \right)}{A_0^2 e^{-2\alpha \eta} e^{-2j\frac{\pi}{\eta} \left( \frac{\eta}{2} \right)}} \right)^{1/2} \]

\[ \ldots \text{(13)} \]
Probability of error for quadrature phase signal is given by:

$$P_{es} = \frac{1}{2} \text{erfc} \left( \frac{\sin^2 \left( \frac{\pi}{2T} \int_0^r g(\tau) d\tau \right)}{A_0^2 e^{-2a r^2} e^{-2 j f \tau} \eta} \right)^{1/2}$$

...(14)

where \(\text{erfc}(u)\) is a complementary error function and it is defined as

$$\text{erfc}(u) = 1 - \text{erf}(u)$$

For slowly fading channel $^{21}$:

$$P_{es} = \frac{1}{2} \text{erfc} \left[ \frac{\alpha' \cos^2 \left( \frac{\pi}{2T} \int_0^r g(\tau) d\tau \right)}{A_0^2 e^{-2a r^2} e^{-2 j f \tau} \eta} \right]^{1/2}$$

...(15)

where $\alpha'$ is a constant.

when $\alpha'$ is random then

$$P_{es} = \int_0^\infty \frac{1}{2} \text{erfc} \left( \sqrt{\alpha' \gamma_b} \right) p(\gamma_b) d\gamma_b$$

...(16)

where $\gamma_b = \frac{S_b}{N_0}$

and $p(\gamma_b) = \frac{1}{\gamma_b} e^{-\gamma_b/\gamma_b}$

...(17)

and $\gamma_b = \frac{E_b}{N_0} E(\alpha^2)$

where $E_b$ is signal energy and $E(\alpha^2)$ is the average value of $\alpha^2$.

Integrating Eq. (16), we get probability of error:

$$P_{ec} = \frac{1}{2} \left( 1 - \frac{\alpha' \gamma_b}{\sqrt{1 + \alpha' \gamma_b}} \right)$$

...(18)

3 Results and Discussion

Signal to noise ratio (SNR) is computed from Eq. (12) and the results are shown in Fig. 1. The variation of signal to noise ratio ($S/N$) with the common design parameter $B_b T$ is shown in Fig. 1. It decreases sharply with $B_b T$ from 0.1 to 0.2 then slowly from 0.2 to 0.3 and then increases between 0.3 to 0.5 and decreases between 0.5 and 0.7 and finally, increases and becomes almost constant. The SNR mainly depends on the value of the integral of $g(t)$ which is inherently depends on the parameter $\beta$ i.e. $B_b T$. The nature of the curve depends on cosine of the integrated value of $g(t)$. $B_b = B_{3dB}/2\pi$ is called normalized 3 dB bandwidth. The larger value of $B_b$ results in a larger signal bandwidth. The values of $E_b N_0$ indicated by the researchers $^7$ for $B_b T_c$, where $T_c = \gamma T_b$ between 0.1 to 0.3 shows that it decreases first and then increases as $B_b T_c$ increases. So, the nature of curve resembles to that of ours.

Power spectral density is numerically computed from Eq.(9) and the result is shown in Fig. 2. The variation of power spectral density with normalized frequency for different values of $B_b T$ is shown in Fig. 2. Our results have been compared with others. The dark continuous line shows our result for $B_b T = 0.7$. The dark broken lines for $B_b T = 0.7$ are due to the researchers $^{1,18}$ and the nature of these curves are same. The power spectral density decreases as normalized frequency increases. Our numerical values of spectral density are close to the values of researchers $^1$. For $B_b T = 1.0$, the results agree with the researchers $^{18}$ as shown by the dotted line. The results for $B_b T = 0.3$ agree with the result as given in Ref. 18 (not shown for simplicity).

Probability of error ($P_{ec}$) is computed from Eq. (13). The result is shown in Fig. 3. The continuous line shows our results and the broken line shows the result as given in Ref. 1. It indicates the variation of probabilities of error with signal to noise ratio for $B_b T= 0.2$ and $B_b T= 0.25$. The probability of error decreases with the increase of SNR and the nature of curves agree with the results as given in Ref. 1. The numerical values of $P_{ec}$ and the SNR are large in comparison with those by researchers $^1$. The reason behind this is that the effect of group delay $t_o$
The probability of error ($P_e$) for GMSK for slowly Rayleigh fading channel is numerically calculated from Eq. (18). The variation of $P_e(\gamma)$ with faded value of signal to noise ratio (SNR) $\gamma_b$ for different channel attenuation constants $\alpha'$ is shown in Fig. 4. Our results have been compared with the results as given in Ref. 1 shown by dotted line for fixed $\alpha' = 0.68$. From Fig.4, it is obvious that for each $\alpha'$ the probability of error decreases with $\gamma_b$.

4 Hardware Implementation

4.1 GMSK modulator

The basic circuit diagram of GMSK modulator with additive Gaussian noise is shown in Fig. 5. In this system, binary data $a(t)$ first of all converted into $(+1, -1)$ sequence and then a stream of rectangular pulses $s(t)$ is achieved. The noise $n_i(t)$ is now added with the signal $s(t)$. The noise corrupted signal is filtered by using Gaussian filter. The filtered output contains $g(t) + n_0(t)$. Now the filtered signal with noise is integrated in the phase accumulator. Thus, the
phase angles $\varphi_s$ and $\varphi_n$ are generated. The quadrature signal components $I$ and $Q$ are obtained from a look up table for the sine and cosine implemented in PLA's. The sine function is generated using two stage approximation technique\textsuperscript{22}. The carrier wave $\cos(\omega_c t)$ is generated by local oscillator and multiplied by in-phase and quadrature phase signals and finally added to get the required modulated signal $y(t)$.

4.2 GMSK detector

The circuit diagram of GMSK detector is shown in Fig. 6. The received signal is multiplied by the same carrier using carrier recovery system using two multipliers, one for in-phase component of signal and other for quadrature phase component of signal.

The higher and other carrier terms are stopped by using low pass filters. The output of filters goes to phase detectors. The output of phase detectors produces the phase angles of in-phase and quadrature components of noise corrupted signals. The phase angles are allowed to go through differentiator and then to the differentiating filter which gives the output of rectangular pulses. The levels of pulses are obtained by envelope detector and finally the binary data is achieved by decision device.

5 Conclusion

In this paper, Gaussian minimum shift keying (GMSK) in additive white Gaussian noise (AWGN) is examined by implementing in-phase I and quadrature phase Q signals obtained from complex phase function. The construction and working of modulator and demodulator has been discussed. The fundamental properties like spectral analysis and bit error rate performance (BER) in presence of AWGN for non-fading and fading channels have been analyzed and studied in detail. The results so obtained, have been compared with others and found to be satisfactory.

References


