Effect of clusterization algorithm with isospin-dependent nuclear charge radii parameterization on azimuthal angle dependence of elliptical flow

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The momentum constraint ($P_{clus}$) in Minimum Spanning Tree (MST) clusterization algorithm has been modified from isospin-independent ($R_{LDM}$) to isospin-dependent ($R_{RR}$) nuclear charge radii parameterization. The influence of MST with modified momentum constraint has been examined on the azimuthal angle dependence of $dN/d (\cos 2\phi)$ (a signature of elliptical flow) for the reaction of $^{50}\text{Ca} + ^{50}\text{Ca}$ and $^{197}\text{Au} + ^{197}\text{Au}$ at incident energies of 50 and 400 MeV/nucleon by using isospin-dependent quantum molecular dynamics (IQMD) model. The study reveals that the influence of modified momentum constraint is significant for lighter colliding nuclei at low incident energies. Furthermore, this effect diminishes as the incident energy and composite system mass increases.

Keywords: Heavy ion collisions, Elliptical flow, Azimuthal angle, Clusterization technique, IQMD model

1 Introduction

Optimization of fragmentation array from the simulations of dynamical model is an essential undertaking in virtue of understanding the physics involved in the overlapping zone of heavy-ion reactions at intermediate energies. The dynamical models simulate the time-evolution of the single particles, its excitation in the compressed zone and de-excitation during the expansion. The phase-space generated by the dynamical models is analyzed by clusterization techniques to identify the different types of fragments so that their properties can be studied. Many clusterization techniques have been proposed in the literature to obtain the best fragmentation array. These techniques can also be used to probe the stability of the fragments generated by the models. The minimum spanning tree (MST)\textsuperscript{1,2} method and its various versions have been used widely in literature. In MST method, two nucleons are assumed to be part of same fragment, if the distance between their centroids fulfills the requirement:

$$|\vec{r}_i - \vec{r}_j| \leq R_{clus}$$ \hspace{1cm} (1)

Here, $\vec{r}_i$ and $\vec{r}_j$ are the co-ordinates of $i^{th}$ and $j^{th}$ nucleon. The $R_{clus}$ is the spatial constraint and its value is taken to be 4 fm. The improved version of the MST algorithm includes additional momentum constraint and dubbed as MSTM (Minimum Spanning Tree with momentum constraint) method\textsuperscript{3}. It neglects those fragments which are close in position and satisfies the Eq. (1), but far in momentum space.

In MSTM algorithm, for a nucleon to be the part of same fragment, firstly their co-ordinate position has to satisfy Eq. 1 and then relative momentum should be less than $P_{clus}$, i.e.,

$$|\vec{p}_i - \vec{p}_j| \leq P_{clus}$$ \hspace{1cm} (2)

Here, $\vec{p}_i$ and $\vec{p}_j$ are the average momentum of $i^{th}$ and $j^{th}$ nucleon. The value of $P_{clus}$ has been considered around the average Fermi momentum ($P_F$) of nucleons.

The fragments identified through these techniques are excited, which can undergo secondary decay in the momentum space. However, when the nuclear matter is dilute, these limitations do not matter. The recent work of Puri and collaborators\textsuperscript{4}, reveals the importance of thermal binding energy cuts in the MST (MST-BT) method over the binding energy cut of cold nuclear matter (MSTB) as well as the MST method in the formation process of fragments. However, at saturation time of the reaction, the multiplicity of various type of fragments obtained via MST-BT is same as that of MST. Therefore, one can conclude that the final stage fragmentation array of MST method does a fair job.
In a study by Zhang et al., isospin dependent spatial constraints in MST method has been developed (called iso-MST) to observe the influence of isospin degree of freedom associated with the nucleons in the fragment production and nuclear stopping. However, the momentum constraint has not been modified in the iso-MST method. Therefore, in present manuscript, the momentum constraint has been modified as per the isospin dependent and independent nuclear charge radii parameterizations, (i.e., $R_{\text{LDM}}$ and $R_{\text{RR}}$) to study the isospin degree of freedom thorough the clusterization technique.

The $R_{\text{LDM}}$ is isospin independent radii parameterization proposed by liquid drop model (LDM) and $R_{\text{RR}}$ is isospin dependent nuclear charge radii parameterization. Both the radii parameterizations read as:

$$R_{\text{LDM}} = r_0 A^{1/3} \quad \cdots (3)$$

Where, $r_0 = 1.12 \text{ fm}$ in present calculations and

$$R_{\text{RR}} = 1.2332 A^{1/3} + (2.8961/A^{2/3}) - 0.18688 A^{1/3} \quad \cdots (4).$$

The above equation includes the isospin dependence through isospin parameter:

$$I = (N-Z)/A \quad \cdots (5)$$

2 IQMD Model

The isospin dependent quantum molecular dynamics (IQMD) model\(^5\) treats different charge states of nucleons, deltas and pions explicitly. In this model, baryons are represented by Gaussian-shaped density distributions, reads as:

$$f_i (r, p, t) = \frac{1}{\pi^{3/2} h^2} e^{-\left(\frac{r_r - r_i}{2L}\right)^2} \left(\frac{1}{2\pi\hbar^2} e^{-\left(\frac{p_r - p_i}{\hbar}\right)^2}\right)^{\frac{2A}{h^2}} \quad \cdots (6)$$

Here, $L$ is the Gaussian width regarded as the interaction range of the particle. The value of $L$ depends upon the mass of the system. For heavier systems, e.g., $^{197}_{79}Au + ^{197}_{79}Au$, $L$=8.66 fm$^2$ and for $^{40}_{20}Ca + ^{40}_{20}Ca$ and lighter nuclei, $L$=4.33 fm$^2$. The nucleons are primarily initialized in a sphere of radius in accordance with the LDM, i.e., isospin independent parameterization. To observe the influence of isospin dependent nuclear charge radii parameterization through cluaterization technique, the $R_{\text{RR}}$ radii parameterization have been incorporated in the model code in place of $R_{\text{LDM}}$. Each nucleon occupies a volume of $h^3$, so that phase space is uniformly filled. The initial momenta are randomly chosen between 0 and Fermi momentum ($p_f$), without any further local constraints. The Fermi momentum ($p_f$) depends on the ground state density.

The successfully initialized nuclei are then boosted towards each other and the hadrons propagate using Hamilton equations of motion:

$$\frac{dr_{ij}}{dt} = \frac{\partial \langle H \rangle}{\partial p_i}; \quad \frac{dp_{ij}}{dt} = -\frac{\partial \langle H \rangle}{\partial r_{ij}} \quad \cdots (7)$$

$$\langle H \rangle = \langle T \rangle + \langle V \rangle \quad \cdots (8)$$

The baryon-baryon potential $V_{ij}$, in the above relation, reads as:

$$V_{ij} = V_{\text{Yukawa}}^{ij} + V_{\text{Skyrme}}^{ij} + V_{\text{dil}}^{ij} + V_{\text{sym}}^{ij} + V_{\text{Coal}}^{ij} \quad \cdots (9)$$

The two nucleons are said to suffer binary collision with each other if the distance between the centroid of their Gaussian wave packets ($r$) is fulfilled by the following requirement.

$$|r_i - r_j| \leq \sqrt{\frac{\sigma_{\text{tot}}}{\pi}}, \sigma_{\text{tot}} = \sigma(\sqrt{s}, \text{type}) \quad \cdots (10)$$

Where, “type” in above equation denotes the in going collision partners (N-N, N-Δ, N-π, etc.). The scattering of nucleons in nuclear matter in low density region can be described in terms of reaction G-matrix. At high energies the influence of pauli blocking is less and kinetic energy is large as compared to the potential. Then the imaginary part of the reaction matrix becomes identical to the transition matrix which describes the scattering between two nucleons. Scattering can be elastic or inelastic.

$$\sigma_{\text{tot}} = \sigma_{\text{el}} + \sigma_{\text{inel}} \quad \cdots (11)$$

The total cross-section is the sum of the elastic and all inelastic cross-sections. The elastic and inelastic cross-sections for proton-proton (pp) and proton-neutron (pn) are used in IQMD model. The neutron-neutron (nn) cross-section is assumed to be equal to pp.

The phase space of nucleons of target as well as projectile nuclei are generated by using IQMD model.
and further this phase space is analysed by using secondary models like MST and MSTM, etc. as discussed in previous section.

3 Results and Discussion

The simulations have been carried out using IQMD model\(^5\) for the reaction of \(^{20}_{50}\)Ca + \(^{20}_{50}\)Ca and \(^{197}_{79}\)Au + \(^{197}_{79}\)Au at incident energies of 50 and 400 MeV/nucleon and scaled impact parameter \(\hat{b} = 0.3\). The reactions are followed till 200 fm/c and clusters are formed with four different algorithms MST (S), MST (S+P), MST (S+isoP) and MST (isoS+isoP) as listed in Table 1. Here ‘S’ corresponds to spatial constraint as in Eq. 1 and ‘P’ corresponds to momentum constraint as in Eq. (2). The isospin dependent spatial constraints (in MST (isoS+isoP)) have been kept as per the optimization made by Zhang et al.\(^4\). The small value of spatial constraints for pp is because of long range Coulomb forces and large value for nn and np is due to properties of neutron-rich nuclei and they do not exert repulsive forces. The calculation of Fermi momentum \(P_F\) has been done through the following formula:

\[
P_F = \eta \frac{P_0}{k_F}.
\]

Where, \(k_F = \left(\frac{3}{2} \Pi^2 \rho\right)^{1/3}\) and \(\rho\) is the normal nuclear matter density defined as:

\[
\rho = \frac{A}{3 \Pi R_{para}^3}.
\]

Here, \(R_{para}\) could be any nuclear charge radii parameterization \(R_{LDM}\) or \(R_{RR}\).

It is worth to mention that the Fermi momentum for \(R_{LDM}\) labelled as \(P_F^{LDM} = 268.4\) MeV/c (the default value used in the IQMD), is system size independent.

However, the \(P_F^{RR}\) is system size dependent. For the mass-symmetric nuclear reactions involving \(^{50}_{20}\)Ca nuclei, the value of \(P_F^{RR}\) is 239.7 MeV/c and for the nuclear reaction involving \(^{197}_{79}\)Au nuclei, \(P_F^{RR}\) is 248.2 MeV/c.

The comparison between MST (S) and MST (S+P) reveals the importance of momentum constraint in addition to spatial constraint; MST (S+P) and MST (S+isoP) reveals significance of using momentum constraint as per isospin nuclear charge radii parameterization. The calculations performed by MST (isoS+isoP) give the significance of using isospin dependent clusterization technique.

Since, the elliptical flow is more influenced by the choice of radii parametrization\(^10\). The elliptical flow is the second harmonic of the Fourier expansion of distribution of particles in azimuthal plane\(^11\) and defined as:

\[
v_2 = <\cos(2 \phi)>
\]

The azimuthal angle \((\phi)\) is defined as the angle between the trajectory of outgoing particles and the reaction plane. Therefore, to observe the influence of different clusterization techniques, the azimuthal angle dependence of \(dN/d(Cos2\phi)\) (which is an indicator of elliptical flow) of FNs (\(A=1\)) has been presented at incident energies of 50 MeV/nucleon (upper panels) and 400 MeV/nucleon (lower panels) for the reactions \(^{20}_{50}\)Ca + \(^{20}_{50}\)Ca in Fig. 1 and \(^{197}_{79}\)Au + \(^{197}_{79}\)Au.

![Fig. 1 – The azimuthal angle dependence of \(dN/d(Cos2\phi)\) of FNs for the reactions \(^{50}_{20}\)Ca + \(^{50}_{20}\)Ca at incident energy of 50 MeV/nucleon (upper panel) and 400 MeV/nucleon (lower panel) for \(R_{LDM}\) (left panels) and \(R_{RR}\) (right panels). Different symbols in figure represent different clusterization techniques mentioned in Table 1.](image)
in Fig. 2 simulated by using $R_{\text{LDM}}$ (left panels) and $R_{\text{RR}}$ (right panels) radii parameterizations.

It is worth notifying that, the calculations performed with MST (S+P) and MST (S+isoP) for $R_{\text{LDM}}$ radii parameterization is exactly same in both figures. It has been observed that $dN/d\cos(2\phi)$ behaves in similar fashion for all cases. Both the figures reveals that the influence of momentum constraint as per radii parameterizations in the clusterization technique is visible for lighter colliding nuclei ($^{20}_{\text{Ca}} + ^{20}_{\text{Ca}}$) at low incident energy (at 50 MeV/nucleon). For heavier colliding nuclei and at relatively higher incident energies, effect of modified momentum cut is very feeble. Hence, role of MST (S+isoP) decreases with increase in system mass as well as incident energy. However, the spatial constraint plays an important role. The magnitude of $dN/d\cos(2\phi)$ decrease equally from $\cos 2\phi = +1$ to -1 as one uses the MST (isoS+isoP) cluster identifier. The influence of MST (isoS+isoP) is more for lighter colliding nuclei at low energies since it considers the momentum constraint effect also.

To further strengthen the interpretation of results, the azimuthal angle distribution of $dN/d\cos(2\phi)$ has been fitted to parabola equation of type: $y = a + bx + cx^2$ (shown by solid black lines in Fig. 1 and 2). Here, $y$ represents $dN/d\cos(2\phi)$ and $x$ represents $\cos 2\phi$. Hence, the equation of parabola fit in both the figures reads as:

$$dN/d\cos(2\phi) = a + b(\cos 2\phi) + c(\cos 2\phi)^2$$

Here, a, b and c are constants. The influence of change in nuclear charge radius from $R_{\text{LDM}}$ to $R_{\text{RR}}$ has been investigated through the radius of curvature (ROC) at $\cos 2\phi = 0$. The values of ROC for the reactions of $^{50}_{\text{Ca}} + ^{50}_{\text{Ca}}$ and $^{197}_{\text{Au}} + ^{197}_{\text{Au}}$ at incident energies of 50 and 400 MeV/nucleon, has been displayed in Table 2. It has been observed that, with increase in nuclear charge radius (as one switches from $R_{\text{LDM}}$ to $R_{\text{RR}}$), the magnitude of $dN/d\cos(2\phi)$ increases in all cases. Moreover, the value of ROC decreases with increase in incident energy for $R_{\text{LDM}}$ while increase for $R_{\text{RR}}$ nuclear charge radii parameterizations.

**4 Conclusions**

The study concludes that the influence of modified momentum constraint in MST clusterization technique has been analysed through isospin dependent nuclear charge radii parameterization on the azimuthal angle distribution of elliptical flow. The study concludes that one should opt for the isospin dependent clusterization technique to optimize the best fragmentation array.

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**References**


**Table 2 – The values of radius of curvature (ROC) obtained from Figs 1 and 2**

<table>
<thead>
<tr>
<th>Clusterization Technique</th>
<th>ROC$_{\text{LDM}}$</th>
<th>ROC$_{\text{RR}}$</th>
<th>ROC$_{\text{LDM}}$</th>
<th>ROC$_{\text{RR}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MST (S)</td>
<td>0.08</td>
<td>0.03</td>
<td>0.029</td>
<td>0.262</td>
</tr>
<tr>
<td>MST (S+P)</td>
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<td>0.029</td>
<td>0.029</td>
<td>0.264</td>
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<tr>
<td>MST (S+isoP)</td>
<td>0.116</td>
<td>0.029</td>
<td>0.029</td>
<td>0.265</td>
</tr>
<tr>
<td>MST(isoS +isoP)</td>
<td>0.0798</td>
<td>0.0304</td>
<td>0.094</td>
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<tr>
<td>MST(isoS +isoP)</td>
<td>0.109</td>
<td>0.0244</td>
<td>0.079</td>
<td></td>
</tr>
</tbody>
</table>

Table 2 – The values of radius of curvature (ROC) obtained from Figs 1 and 2 for all the reactions of $^{50}_{\text{Ca}} + ^{50}_{\text{Ca}}$ and $^{197}_{\text{Au}} + ^{197}_{\text{Au}}$.