Stochastic Multifacility Location Problem under Triangular Area Constraint with Squared Euclidean Norm

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This investigation is the extension of our previous work in which it is required to find the locations of a number of new facilities in a prescribed triangular area constraint with respect to a given number of existing facilities where the weights considered in the objective function are the random variables with discrete probabilities and the distance between the facilities is Squared Euclidean. The stochastic multifacility location problem under Squared Euclidean norm with triangular area constraint has been formulated and solved by using Kuhn-Tucker conditions. Numerical example has also been solved by using the proposed method. Thus the outcome of the present work is a new method of finding the solution of a constrained stochastic multifacility location problem with Squared Euclidean.

Keywords: Multifacility Location, Stochastic, Triangular Area Constraint, Squared Euclidean, Kuhn-Tucker Conditions.

Introduction

The facility location problem with Squared Euclidean norm is of great importance in various realistic situations. For example, there exist emergency service location problems such as hospitals, fire stations, etc. in which costs increase quadratically instead of linearly with distance. This necessity in tackling these sorts of problems has motivated us to study the stochastic version of the multifacility location problem with Squared Euclidean norm. Investigations in the past have mainly been done without taking into account the availability of the area into which the new locations fall. Santra and Nasira have considered multifacility location problem under triangular and semi-open rectangular area constraints with Euclidean norm. Santra has considered the stochastic version of the problem under triangular and semi-open rectangular area constraints with Euclidean norm. Santra has considered the deterministic as well as stochastic multifacility location problem with circular area constraint. The present study is the extension of our previous work in which the weights considered are probabilistic in nature and the distance between the facilities are Squared Euclidean.

Problem formulation and solution procedures

The stochastic multifacility location problem considered with triangular area constraint can be stated as:

\[
\text{Minimize } f ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) = \sum_{1 \leq i < k \leq n} E(v_{jk}) [(x_j - x_k)^2 + (y_j - y_k)^2] + \sum_{i=1}^{m} \sum_{j=1}^{n} E(w_{ij}) [(x_j - \bar{x}_i)^2 + (y_j - \bar{y}_i)^2] \quad \ldots (1)
\]

Subject to \( ax_1 + by_1 + c \leq 0 \) (\( a > 0, b > 0, c < 0 \)) \ldots (2)

and \( x_j \geq 0, y_j \geq 0 \) (\( j = 1, 2, \ldots, n \)), where all the parameters are defined as per Santra.

In the present paper we have considered the general multifacility location problem in which the interactions among new facilities as well as between new and existing facilities are considered. From the structure of the objective function it is clear that the optimization over \( x_j \) and \( y_j \) can be done separately. Here the objective function is quadratic and the side constraint is linear. We get feasible solution either corresponding to the interior points or else occur on the boundary only. We use Kuhn-Tucker theory to get the solution of the problem for which we construct the auxiliary function as follows:

\[
h(x,y) = f(x,y) - \lambda [(ax + by + c) + \eta^2]
\]

i.e.,

\[
h((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) = f ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) - \sum_{j=1}^{n} \lambda_j [(ax_j + by_j + c) + \eta_j^2] \quad \ldots (3)
\]
where $\lambda_j \geq 0$ and $\eta_j^2$ are the artificial variables as given by Santra\textsuperscript{a}.

Now by using Kuhn-Tucker theory we will be getting the set of necessary conditions as $\frac{\partial h}{\partial x_j} = 0$;

$$\frac{\partial h}{\partial y_j} = 0 \quad \text{where} \quad \lambda_j (ax_j + by_j + c) = 0, \quad \text{where} \quad \lambda_j \geq 0.$$

Thus, we get:

$$\frac{\partial h}{\partial x_i} = \frac{\partial f}{\partial x_i} - a\lambda_i = 0 \quad \text{... (4)}$$

$$\frac{\partial h}{\partial y_j} = \frac{\partial f}{\partial y_j} - b\lambda_i = 0 \quad \text{... (5)}$$

We may mention that the objective function, $f$, given by (1) here as can be verified from the Hessian Matrix associated with it, is convex and hence the set of necessary conditions become sufficient for optimality.

Now $\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial y_j}$ are given as:

$$\frac{\partial f}{\partial x_i} = 2 \sum_{k=1}^{n} E (v_{jk})(x_i - x_k) + 2 \sum_{l=1}^{m} E(w_{ji}) (x_i - \tilde{x}_i), \quad \text{... (6)}$$

$$\frac{\partial f}{\partial y_j} = 2 \sum_{k=1}^{n} E (v'_{jk})(y_j - y_k) + 2 \sum_{l=1}^{m} E(w_{ji}) (y_j - \tilde{y}_j), \quad \text{... (7)}$$

in which $v_{jk} = \begin{cases} v_{ik}, & k > j \\ v_{ij}, & k < j \end{cases}$

We may note here that the necessary conditions for the occurrence of the minimum are also sufficient in view of the convexity of the objective function. We are thus assured of global minimum in all possible cases, viz. $\lambda_j = 0$ or $\lambda_j \neq 0$. The case $\lambda_j = 0$ corresponds to the unconstrained problem. However, the constraint condition (2) has to be fulfilled by the solution for its feasibility. If this solution is not feasible, we are left with the case $\lambda_j \neq 0$ only; that is, the boundary is operative. This means the optimum now is attained on the boundary only. Hence, we are to examine only two possible cases, viz., (i) when $\lambda_j = 0$ and (ii) when $\lambda_j \neq 0$ ($j = 1, 2, \ldots, n$). First we consider the case when $\lambda_j = 0$.

Case – I: $\lambda_j = 0 \quad (j = 1, 2 \ldots n)$

Since $\lambda_j = 0$, we get $\frac{\partial f}{\partial x_j} = 0$ and $\frac{\partial f}{\partial y_j} = 0$ and from equations (6) and (7), a straight forward calculation will give the linear expressions for $x_j$ and $y_j$ as:

$$x_j = \frac{\sum_{k=1}^{n} E(v'_{jk})x_k + \sum_{l=1}^{m} E(w_{ji})\tilde{x}_i - \frac{ab}{a^2 + b^2} \sum_{l=1}^{m} E(w_{ji})\tilde{y}_i - \frac{ac}{a^2 + b^2} \sum_{l=1}^{m} E(w_{ji})}{\sum_{k=1}^{n} E(v'_{jk}) + \sum_{l=1}^{m} E(w_{ji})}, \quad j = 0, 1, 2, \ldots, n \quad \text{... (8)}$$

The equation (8) represents n linear equations from which the $x$-coordinates, $x_j$ ($j=1,2,\ldots,n$), of n new facilities are to be determined and the equation (9) represents n linear equations from which the $y$-coordinates, $y_j$ ($j=1,2,\ldots,n$), of n new facilities are to be determined. Now the solution $(x_j, y_j)$ obtained by the above method has to be tested whether it satisfies the constraint (2). If it satisfies (2), the problem is solved and $(x_j, y_j); j = 1, 2, \ldots, n$ gives the optimum location for the new facilities sought. If $(x_j, y_j)$ do not satisfy (2), we have the only alternative of considering the case when $\lambda_j \neq 0 \quad (j = 1, 2, \ldots, n)$ which makes all the new facilities lie on the boundary.

Case – II: $\lambda_j \neq 0 \quad (j = 1, 2, \ldots, n)$

Since $\lambda_j \neq 0$, from one of the equations of necessary conditions we get $y_j = \frac{c}{b} - \frac{a}{b} x_j$ and from the other two we get $\frac{\partial f}{\partial x_j} = \frac{a}{b} \frac{\partial f}{\partial y_j}$, After substituting the expression for $\frac{\partial f}{\partial x_j}, \frac{\partial f}{\partial y_j}$ and $y_j$ in this equation, a straight forward calculation will lead to the linear expression for $x_j$ as given below:

The equation (10) represents n linear equations from which the $x$-coordinates, $x_j$ ($j=1,2,\ldots,n$), of n new facilities are to be determined. It may be pointed out that after finding the values of all $x_j (j = 1, 2, \ldots, n)$ by solving the set of linear equations, one need not calculate the values of $y_j$ by using the same method as these can be determined directly with the help of the determined values of $x_j$. The equation (10) represents n linear equations from which the $x$-coordinates, $x_j$ ($j=1,2,\ldots,n$), of n new facilities are to be determined and the equation (9) represents n linear equations from which the $y$-coordinates, $y_j$ ($j=1,2,\ldots,n$), of n new facilities are to be determined. Now the solution $(x_j, y_j)$ obtained by the above method has to be tested whether it satisfies the constraint (2). If it satisfies (2), the problem is solved and $(x_j, y_j); j = 1, 2, \ldots, n$ gives the optimum location for the new facilities sought. If $(x_j, y_j)$ do not satisfy (2), we have the only alternative of considering the case when $\lambda_j \neq 0 \quad (j = 1, 2, \ldots, n)$ which makes all the new facilities lie on the boundary.

$$x_j = \frac{\sum_{k=1}^{n} E(v'_{jk})x_k + \sum_{l=1}^{m} E(w_{ji})\tilde{x}_i - \frac{ab}{a^2 + b^2} \sum_{l=1}^{m} E(w_{ji})\tilde{y}_i - \frac{ac}{a^2 + b^2} \sum_{l=1}^{m} E(w_{ji})}{\sum_{k=1}^{n} E(v'_{jk}) + \sum_{l=1}^{m} E(w_{ji})}, \quad j = 0, 1, 2, \ldots, n \quad \text{... (10)}$$
Numerical Example

For illustration of the solution procedure of the problem presented in previous section, let us consider the example involving 3 new facilities and 2 existing facilities where the new facilities are to be located in a restricted triangular area given by \(x_1 + y_1 - 8 \leq 0\). Let the co-ordinates of the 2 existing facilities be \((\hat{x}_1, \hat{y}_1) = (4, 8)\) and \((\hat{x}_2, \hat{y}_2) = (2, 4)\). Let the cost per unit distance among new facilities and the corresponding probabilities be given as follows:

\[
v_{121} = 6, p_{121} = \frac{1}{3}; \quad v_{122} = 3, p_{122} = \frac{2}{3}; \quad v_{131} = 8, p_{131} = \frac{1}{4}; \quad v_{132} = 4, p_{132} = \frac{3}{4}; \quad v_{231} = 3, p_{231} = \frac{1}{5}; \quad v_{232} = 4, p_{232} = \frac{1}{6}; \quad v_{233} = 6, p_{233} = \frac{1}{6}.
\]

Let the cost per unit distance between new and existing facilities and the corresponding probabilities be given as follows:

\[
w_{111} = 4, q_{111} = \frac{1}{4}; \quad w_{112} = 8, q_{112} = \frac{3}{4}; \quad w_{121} = 6, q_{121} = \frac{1}{3}; \quad w_{122} = 2, q_{122} = \frac{1}{2}; \quad w_{123} = 12, q_{123} = \frac{1}{6}; \quad w_{211} = 6, q_{211} = \frac{5}{6}; \quad w_{212} = 12, q_{212} = \frac{1}{6}; \quad w_{221} = 7, q_{221} = \frac{1}{7}; \quad w_{222} = 14, q_{222} = \frac{6}{7}; \quad w_{311} = 3, q_{311} = \frac{2}{3}; \quad w_{312} = 6, q_{312} = \frac{1}{5}; \quad w_{321} = 12, q_{321} = \frac{1}{2}; \quad w_{322} = 2, q_{322} = \frac{1}{2}; \quad w_{323} = 3, q_{323} = \frac{1}{3}.
\]

Case – I A: \(\lambda_j = 0\) (j = 1, 2, 3)

With the above data, we formulate the set of linear equations as described in Case – I and after solving the linear equations we get the \(x\) co-ordinates as 2.92, 2.78, 2.98 and the corresponding \(y\) co-ordinates as 5.84, 5.56, 5.96. Thus the co-ordinates of the 3 new locations are (2.47, 5.52), (2.60, 5.39) and (2.51, 5.48).

Case – IIA: \(\lambda_j \neq 0\) (j = 1, 2, 3)

With the above data, we formulate the set of linear equations as described in Case – II and after solving the linear equations we get the \(x\) co-ordinates as 2.47, 2.60, 2.51 and the corresponding \(y\) co-ordinates as 5.52, 5.39, 5.48. Thus the co-ordinates of the 3 new locations are (2.47, 5.52), (2.60, 5.39) and (2.51, 5.48).

Conclusion

Many researchers have worked on multifacility location problems by considering the constraints over the capacity of the locations but nobody has considered the area constraint of the type which has been considered in the present paper. It might likewise be specified that the work containing stochastic contemplations has additionally gotten less consideration despite the fact that practical nature of the issues requests probabilistic examinations. The present work is the extension of our previous work\(^4\)\(^6\).

Physically such probabilistic approach is of great importance in the sense that the values of weights from various origins (sources) to different destinations are not the fixed quantities but they take random values in different situations. The facility location problem with Squared Euclidean norm is of great importance in various realistic situations. For example, there exist emergency service location problems such as hospitals, fire stations, etc. in which costs increase quadratically instead of linearly with distance. This necessity in tackling these sorts of problems has motivated us to study the stochastic version of the multifacility location problem with Squared Euclidean norm. Hence, the present work is new in relation with the existing works.

References


