Incorporating multi-delivery policy and quality assurance into economic production lot size problem

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This paper incorporates a multi-delivery policy and quality assurance into an imperfect economic production quantity (EPQ) model with scrap and rework. A portion of nonconforming items produced is considered to be scrap, while other is assumed to be repairable and is reworked in each cycle when regular production ends. Finished items can only be delivered to customers if whole lot is quality assured after rework. Fixed quantity multiple installments of finished batch are delivered by request to customers at a fixed interval of time. Expected integrated cost function per unit time is derived. A closed-form optimal batch size solution to the problem is obtained. Two special cases are examined and a numerical example given to demonstrate model’s practical usage.

Keywords: EPQ model, Lot sizing, Multiple deliveries, Production, Rework, Scrap

Introduction

Economic order quantity (EOQ) model¹ employs mathematical techniques to balance setup and inventory holding costs, and derives an optimal order size to assist corporations in minimizing long-run average inventory costs. In manufacturing sector, when products are produced in-house instead of being acquired from outside suppliers, economic production quantity (EPQ) model is often utilized to cope with finite production-inventory replenishment rate in order to minimize total costs per unit time²-⁴. Classic EPQ model assumes that all items produced are of perfect quality. However, in real-life production systems, due to process deterioration and/or other factors, generation of imperfect quality items is inevitable. Studies have been carried out to enhance classic EPQ model by addressing the issue of defective items produced⁵-¹⁴. Defective items sometimes can be reworked and repaired, and hence overall production costs can be significantly reduced¹⁵-²³. Hayek & Salameh¹⁶ assumed that all defective items produced are repairable and derived an optimal operating policy for an EPQ model under the effect of rework of all defective items. Jamal et al¹⁸ studied optimal manufacturing batch size with rework process at a single-stage production system. Chiu & Chiu¹⁹ proposed mathematical modeling for determining optimal batch size and backordering level for an imperfect finite production rate model with backlogging and failure in repair.

Another unrealistic assumption of classic EPQ model is continuous inventory issuing policy for satisfying product demand. Goyal²⁴ considered integrated inventory model for a single supplier-single customer problem. Studies have since been carried out to address various aspects of supply chain optimization²⁵-³⁵. Golhar & Sarker²⁶ developed a simple algorithm to compute optimal batch size for a system where a just-in-time (JIT) buyer demands frequent deliveries of small lots of certain products. Viswanathan³⁰ reexamined integrated vendor -buyer inventory models. Sarker & Khan¹¹ considered a manufacturing system that procures raw materials from suppliers in a lot and processes them into finished products, which are delivered to outside buyers at fixed points in time. Ouyang et al³⁴ studied single-vendor single-buyer integrated production inventory models with stochastic demand in controllable lead time.

This paper investigates joint effect of multi-delivery policy, scrap, and rework on optimal replenishment lot size of EPQ model.
This paper incorporates a multi-delivery policy and quality assurance into an imperfect EPQ model with scrap and rework. Consider that during regular production time, an \( x \) portion of defective items is produced randomly, at a production rate \( d \). Among defective items, a \( \theta \) portion is assumed to be scrap and rest can be reworked and repaired at a rate \( P_1 \), in each cycle after a production run. For regular supply, constant production rate \( P \) has to be larger than sum of demand rate \( \lambda \) and production rate \( d \). Thus, \( (P-d-\lambda)>0 \) or \( (1-x-\lambda/P)>0 \); where \( d=P_x \).

It is assumed that finished items can only be delivered to customers if whole lot is quality assured at the end of rework. Fixed quantity of \( n \) installments of finished batch is delivered by request to customers, at a fixed interval of time during production downtime \( t \) (Fig. 1). Cost parameters considered in proposed model include setup cost \( K \), unit holding cost \( h \), unit production cost \( C \), disposal cost per scrap item \( C_s \), unit rework cost \( C_r \), holding cost \( h_1 \) for each reworked item, fixed delivery cost \( K_1 \) per shipment, and delivery cost \( C_T \) per item shipped to customers. Fig. 1 gives \[^{16,19}\]

\[
T = t_1 + t_2 + t_3 \quad \ldots(1)
\]

\[
t_1 = \frac{Q}{P} = \frac{H_1}{P-d} \quad \ldots(2)
\]

\[
t_2 = \frac{xQ(1-\theta)}{P_1} \quad \ldots(3)
\]

\[
t_3 = nt_n = T - (t_1 + t_2) = Q \left( \frac{(1-\theta)x}{\lambda} - \frac{1}{P} - \frac{x(1-\theta)}{P_1} \right) \quad \ldots(4)
\]

\[
H_1 = (P-d)t_1 = (P-d)\frac{Q}{P} = (1-x)Q \quad \ldots(5)
\]

\[
H = H_1 + P_1t_2 = Q(1-\theta x) \quad \ldots(6)
\]

where, \( T \), cycle length; \( H \), maximum level of on-hand inventory when regular production process ends; \( H_1 \), maximum level of on-hand inventory when rework process finishes; \( Q \), production lot size; \( t_1 \), production uptime for proposed EPQ model; \( t_2 \), time for reworking of defective items; \( t_3 \), time for delivering all quality assured finished products; \( t_n \), fixed interval of time between each installment of finished products delivered during \( t_3 \); \( l(t) \), on-hand inventory of perfect quality items at time \( t \); and \( l_d(t) \), on-hand inventory of defective items at time \( t \).

On-hand inventory of defective items during production uptime \( t_1 \) and reworking time \( t_2 \) (Fig. 2) shows that maximum level of on-hand defective items is \( dt_1 \) and

\[
dt_1 = Pxt_1 = xQ. \quad \ldots(7)
\]

A \( q \) portion among nonconforming items is assumed to be scrap [Eq. (8)]. Other repairable portion (1-\( l \)) is reworked right after production uptime \( t_1 \) ends.

\[
\theta dt_1 = \theta Pxt_1 = \theta xQ \quad \ldots(8)
\]

Total costs per cycle \( TC(Q) \) consists of setup cost, variable production cost, variable rework cost, disposal cost, fixed and variable delivery cost, holding cost during \( t_1 \) and \( t_2 \), variable holding cost for items reworked, and holding cost for finished goods during delivery time \( t_3 \) where \( n \) fixed-quantity installments of finished batch are delivered by request to customers at a fixed interval of time. Cost for each delivery is

\[
K_1 + C_T \left( \frac{H}{n} \right) \quad \ldots(9)
\]

Total delivery costs for \( n \) shipments in a cycle are

\[
n \left[ K_1 + C_T \left( \frac{H}{n} \right) \right] = nK_1 + C_T H = nK_1 + C_T Q(1-\theta x) \quad \ldots (10)
\]
Eq. (14) is resulting positive provided that $K, n, K_1, \lambda, Q_1$ and $(1-\theta E[x])$ are all positive. Second derivative of $E[TCU(Q)]$ with respect to $Q$ [Eq. (14)] is greater than zero, and hence $E[TCU(Q)]$ is a convex function for all $Q$ different from zero. Optimal production lot size $Q^*$ can be obtained by setting first derivative [Eq. (13)] of $E[TCU(Q)]$ equal to zero.

After rearrangement, one obtains

$$
\frac{d^2 E[TCU(Q)]}{dQ^2} = \frac{2(K+nK_1)\lambda}{Q^3(1-\theta E[x])} \quad \ldots (14)
$$

**Convexity of $E[TCU(Q)]$**

Optimal production lot size can be obtained by minimizing expected cost function $E[TCU(Q)]$. Differentiating $E[TCU(Q)]$ with respect to $Q$ gives first and second derivative as

$$
\frac{dE[TCU(Q)]}{dQ} = \frac{K\lambda}{Q^2(1-\theta E[x])} - \frac{nK_1\lambda}{Q^2(1-\theta E[x])} + \frac{h\lambda}{2P(1-\theta E[x])}
$$

$$
+ \frac{h\lambda}{2P(1-\theta E[x])} \left[ \frac{2E[x]-E[x] - \theta(E[x])}{Q(1-\theta E[x])} \right] (1-\theta)
$$

$$
+ \left( \frac{n-1}{n} \right) \left[ \frac{h(1-\theta E[x])}{2} + \frac{h\lambda}{2P} \frac{E[x](1-\theta)^2}{Q(1-\theta E[x])} \right]
$$

$$
\ldots (13)
$$

and

$$
Q^* = \sqrt{\frac{2(K+nK_1)\lambda}{P^2 h_\lambda + \frac{2h\lambda}{P} \frac{2E[x]-E[x]-\theta(E[x])}{Q(1-\theta E[x])}}}
$$

$$
+ \left( \frac{n-1}{n} \right) \left[ \frac{h(1-\theta E[x])}{2} + \frac{h\lambda}{2P} \frac{E[x](1-\theta)^2}{Q(1-\theta E[x])} \right]
$$

$$
\ldots (17)
$$

**Special Cases**

**Case 1: when $\theta=1$**

Suppose that all nonconforming items are scrap. On-hand inventory of perfect quality items is illustrated in Fig. 3. Let $TC(Q)$ be total production-inventory-delivery costs per cycle when all defective items are scrap, then

$$
I_d(t) = \theta \text{ portion of defective items are scrap}
$$

Fig. 2—On-hand inventory of defective items in EPQ model with multi-delivery policy, scrap, and rework
\[ TC_1(Q) = CQ + K + C_1(Q) + nK + \left[ \frac{H + dh}{2} \right] + \left[ \frac{n - 1}{2n} \right] Ht. \] ...(18)

Expected production-inventory-delivery cost per unit time for this special model is

\[ E[TCU_1(Q)] = \frac{E[TCU_1(Q)]}{E[T]} = C_\lambda \left[ \frac{(K + nK)}{Q} \right] + C_\lambda + hQ + \frac{hQ}{2} \frac{hQ}{2P} \] ...(24)

First and the second derivatives of \( E[TCU_1(Q)] \) are shown in Eqs (25) and (26).

\[ \frac{dE[TCU_1(Q)]}{dQ} = \frac{K\lambda}{Q(1 - E[x])} - \frac{nK\lambda}{Q(1 - E[x])} + \frac{h\lambda}{2P(1 - E[x])} \] ...(20)

\[ \frac{d^2 E[TCU_1(Q)]}{dQ^2} = 2 \left( \frac{K + nK_1}{Q^2} \right) \] ...(21)

Since Eq.(21) is positive, \( E[TCU_1(Q)] \) is a convex function for all \( Q \) different from zero. By setting first derivative of \( E[TCU_1(Q)] \) = 0 gives optimal production batch size.

\[ Q^* = \frac{2(K + nK_1)\lambda}{h\lambda + \left( \frac{n - 1}{n} \right) h(1 - E[x]) - h(\frac{\lambda}{P})(1 - E[x])} \] ...(22)

Case 2: when \( x = 0 \)

When all items produced are of perfect quality, proposed model becomes classic EPQ model with multi-delivery policy. On-hand inventory of perfect quality item is similar (Fig. 3), except the slope = \( P \) instead of \( P - d \) during production uptime \( t \). Let \( TC_2(Q) \) denote total production-inventory-delivery per cycle when no defective items produced, then

\[ TC_2(Q) = CQ + K + C_1Q + nK + \left[ \frac{H}{2} \right] + \left[ \frac{n - 1}{2n} \right] Ht. \] ...(23)

Expected production-inventory-delivery cost per unit time for this special model is

\[ E[TCU_2(Q)] = \frac{TCU_2(Q)}{T} = C_\lambda \left[ \frac{(K + nK)}{Q} \right] + C_\lambda + \frac{hQ}{2P} \] ...(24)
Thus $E[TCU_2(Q)]$ is a convex function and optimal batch size can be derived as

$$Q^* = \sqrt{\frac{2(K + nK_1)\lambda}{h\lambda + (n-1)h\left(1 - \frac{\lambda}{P}\right)}}$$  \hspace{1cm} (27)

**Numerical Example**

Suppose that a product can be manufactured at a rate of 60,000 units per year and this item has experienced a flat demand rate of 3,400 units per year. During production uptime, random defective rate is assumed to be uniformly distributed over the interval [0, 0.3]. Among defective items, a portion $\theta = 0.1$ is considered to be scrap and other portion can be reworked and repaired, at a rate $P_1 = 2,100$ units per year. Additional parameters considered by this example are given as follows: $C_S = $60 per item reworked; $C_r = $20 per scrap item; $C = $100 per item; $K = $20,000 per production run; $h = $20 per item per year; $h_1 = $40 per item reworked per unit time (year); $n=4$ installments of finished batch are delivered per cycle; $K_1 = $4,350 per shipment, a fixed cost; and $C_t = $0.1 per item delivered.

Optimal batch size $Q^* = 3495$ can be obtained from Eq. (17) and long-run average production-inventory-delivery costs per year $E[TCU(Q)] = $448,390 from Eq. (12). Effect of variation of batch size on overall cost function $E[TCU(Q)]$ and on various components of $E[TCU(Q)]$ are illustrated in Fig. 4. As $E[x]$ increases, both holding cost (during $t_1$) and total delivery costs increase significantly (Fig. 6). Optimal batch size $Q^* = 4755$ and long-run average cost $E[TCU(Q^*)] = $475,264 for special case 1 (situation when all nonconforming items are scrap) can be calculated using Eqs (22) and (19). So do numerical solutions for special case 2 (situation when all items produced are of perfect quality): $Q^* = 4079$ and $E[TCU_2(Q^*)] = $402,685 [Eqs (27) and (24)].

**Conclusions**

Classic EPQ model assumes a continuous issuing policy for satisfying product demand and all items produced are of perfect quality. In real life situation, a multi-delivery policy is used practically in lieu of continuous issuing policy, and it is inevitable to generate defective items during production run. This study incorporates multiple installments of finished batch delivery policy and quality assurance into an imperfect economic production quantity (EPQ) model with scrap and rework. Mathematical modeling is used and expected integrated production-inventory-delivery cost per unit time is derived. Convexity of integrated cost function is proved, and a closed-form optimal batch size solution to the problem is obtained.

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Appendix A

Computation of holding cost of finished products during delivery time $t_3$ [very last term in Eq.(11)]

(1) When $n=1$, total holding cost in delivery time $t_3=0$.

(2) When $n=2$, total holding costs in delivery time $t_3$ become (Fig.A-1):

$$h\left(\frac{H}{2} \times \frac{t_3}{2}\right) = h\left(\frac{1}{2^2}\right)Ht_3$$  \hspace{1cm} ...(A-1)

![Fig. A-1 — Average on-hand inventory of finished products during delivery time $t_3$ when $n=2$, in EPQ model with multi-delivery policy](image)

(3) When $n=3$, total holding costs in delivery time $t_3$ become (Fig.A-2):

$$h\left(\frac{2H}{3} \times \frac{t_3}{3} + \frac{1H}{3} \times \frac{t_3}{3}\right) = h\left(\frac{2+1}{3^2}\right)Ht_3$$  \hspace{1cm} ...(A-2)

![Fig. A-2 — Average on-hand inventory of finished products during delivery time $t_3$ when $n=3$, in EPQ model with multi-delivery policy](image)

(4) When $n=4$, total holding costs in delivery time $t_3$ become:

$$h\left(\frac{3H}{4} \times \frac{t_3}{4} + \frac{2H}{4} \times \frac{t_3}{4} + \frac{1H}{4} \times \frac{t_3}{4}\right) = h\left(\frac{3+2+1}{4^2}\right)Ht_3$$  \hspace{1cm} ...(A-3)

Therefore, following general term (Fig.A-3) for total holding costs during delivery time $t_3$ can be obtained:

$$h\left(\frac{1}{n^2}\right)\left[\sum_{i=1}^{n-1} i\right]Ht_3 = h\left(\frac{1}{n^2}\right)\left[\frac{n(n-1)}{2}\right]Ht_3 = h\left(\frac{n-1}{2n}\right)Ht_3$$  \hspace{1cm} ...(A-4)
Appendix B

Computation of Eq. (12), Recall Eq. (11) as follows:

\[
TC (Q) = CQ + K + C_R \left[ x (1-\theta) Q \right] + C_S \left[ x \theta Q \right] + C_T \left[ Q (1-\theta x) \right] + nK_1 \\
+ \frac{h_1 \cdot P_1 \cdot t_2}{2} \cdot (t_2) + h \left[ H_t + dt_1 \left( t_1 \right) + H_t + H \left( t_2 \right) \right] + h \left( \frac{n-1}{2n} \right) Ht_3
\]

\[\text{... (11)}\]

Then

\[
TC (Q) = CQ + K + nK_1 + C_R \left[ x (1-\theta) Q \right] + C_S \left[ x \theta Q \right] + C_T \left[ Q (1-\theta x) \right] \\
+ \frac{hQ^2}{2P} + \frac{hQ^2}{2P_1} \left[ (2x - x^2 - \theta x^2)(1-\theta) \right] \\
+ \left( \frac{n-1}{n} \right) \left[ \frac{hQ^2 (1-\theta x)^2}{2\lambda} - \frac{hQ^2 (1-\theta x)}{2P} - \frac{hQ^2 x (1-\theta)(1-\theta x)}{2P_1} \right] + \frac{h_x^2 Q^2 (1-\theta)^2}{2P_1}
\]

\[\text{... (B-1)}\]

and

\[T = \frac{Q}{\lambda} (1-\theta x).
\]

\[\text{... (B-2)}\]

Since

\[E[TCU (Q)] = \frac{E[TC (Q)]}{E[T]}
\]

\[\text{... (B-3)}\]

then,

\[
E[TCU (Q)] = \frac{C \lambda}{1-\theta E[x]} + \frac{(K + nK_1) \lambda}{Q (1-\theta E[x])} + \frac{C_R E[x] (1-\theta) \lambda}{(1-\theta E[x])} + \frac{C_S E[x] \theta \lambda}{(1-\theta E[x])} \\
+ C_T \lambda + \frac{hQ \lambda}{2P (1-\theta E[x])} + \frac{hQ \lambda}{2P_1 (1-\theta E[x])} \left[ (2E[x] - (E[x])^2 - \theta (E[x])^2)(1-\theta) \right] \\
+ \frac{n-1}{n} \left[ \frac{hQ (1-\theta E[x])}{2} - \frac{hQ \lambda}{2P} - \frac{hQ E[x] (1-\theta) \lambda}{2P_1} \right] + \frac{h_1 (E[x])^2 Q \lambda (1-\theta)^2}{2P_1 (1-\theta E[x])}
\]

\[\text{... (12)}\]
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