Short Communication

Redefining significant wave periods for modelling and prediction

N.Unnikrishnan Nair¹, G. Muraleedharan² & P.G. Kurup²*

¹Department of Statistics, Cochin University of Science and Technology, Cochin 682 022, Kerala, India
²Department of Physical Oceanography, Cochin University of Science and Technology, Fine Arts Avenue, Cochin 682 016, Kerala, India
*E-mail: kuruppg@yahoo.co.uk

Received 24 September 2002, revised 28 June 2003

Significant wave periods are redefined and a modified Erlang distribution model is suggested for the redefined significant wave periods by the method of characteristic function. Various prediction formulae have also been derived from this model for the redefined significant wave period statistics and compared with actual values indicating that the estimated values have acceptable accuracy.

Key words: Significant wave approach, random wave, concept, modelling, redefined significant wave period, prediction

The significant wave approach (SWA) and the random wave concept (RWA) which are the tools for coastal engineering activities, can be made more effective by redefining the significant wave and suggesting appropriate models and prediction formulae for various wave parameters. An attempt is made here in this aspect. Significant and random wave concepts provide the usual methods of tackling problems in ocean engineering activities. Significant wave approach was put forward by Sverdrup & Munk to represent the random wave field. Kinsman opined that the significant wave approach offers only a statistical summary to deal with the unmanageable large number of actual waves. It is unrealisable as the number of waves (n) is undefined. It is in terms of this statistic that the Sverdrup-Munk Theory does its forecasting. Significant wave height was redefined and Weibull Model was suggested for simulating its frequency distribution by the method of characteristic function. Wave period is another important parameter determining operational conditions for all ocean engineering activities. Putz was the first to suggest that wave periods follow a gamma-type distribution. Muraleedharan et al. fitted visual data on wave periods off Valiathura and Mangalore on the south-west coast of India to Gamma, Rayleigh, exponential and Brestschneider models and found the Gamma distribution to be better than the other competing models.

The present study considers the Gamma model as the parent distribution for zero up-crossing ocean wave periods. The significant wave period is redefined and a modified Gamma model is suggested as a sampling distribution model for the redefined significant wave period statistic by the method of characteristic function. Parametric relations have also been derived from this modified model for estimating various redefined wave parameters.

To suggest Gamma model for wave period distribution, we assume that ‘T’ is the random variable representing wave periods and ‘t’ is a particular value of T; ‘T’ is non-negative. Let N(t) be the number of waves of period ‘t’ in a location at a given point of time satisfying

1) \(\frac{dN(t)}{dt} \propto N(t)\), i.e. the change in the number of waves of period ‘t’ is proportional to the actual number of waves of the same period.

2) The change in N(t) depends on the wave period. i.e. the proportionality between N(t) and \(\frac{dN(t)}{dt}\) is a variable quantity which changes inversely with the wave period.

3) When the wave period is small (large) the rate of change is also small (large).

However these changes cannot exceed a fixed quantity. These conditions are met if we choose

\(\frac{dN(t)}{dt} = ((k/\beta) - 1/\beta); \quad N(t), t, k, \beta > 0\).

Writing \(k = \alpha - 1\),

\(N(t) = \text{const.} \exp - (t/\beta) . t^{\alpha - 1}\)

If ‘N’ is the total number of waves at the given point of time, the probability density of ‘t’ is

\(f(t) = \frac{N(t)}{N} = \text{const.} . \exp - (t/\beta) . t^{\alpha - 1}\)

\([1/(\beta^{\alpha} \Gamma (\alpha)]) . \exp - (t/\beta) . t^{\alpha - 1}\)
This gives the gamma density function. $\beta = \sigma^2/\bar{T}$, $\alpha = (\bar{T}/\sigma)^2$, $\bar{T}$ = mean wave period and $\sigma$ = S.D which are determined from the data. $\alpha$ and $\beta$ are the shape and scale parameters respectively. The distribution function of Gamma model (incomplete Gamma functions) can be obtained from incomplete Gamma function tables.

Power series expansion for this incomplete Gamma function is:

$$F(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta)} \Gamma(\alpha + \beta) \exp(-\lambda t)$$

Since the various wave period prediction formulae derived from these are complicated and a decimal point accuracy is not required in the case of wave periods, the $\alpha$ values are approximated to the nearest integer to obtain Erlang distribution model and prediction formulae.

The density function of Gamma is given as:

$$f(t) = [\alpha \cdot \Gamma(\alpha)] (\lambda t)^{\alpha-1} / \alpha ! e^{-\lambda t} \; ; \alpha > 1$$

where ‘$\alpha$’ and ‘$\lambda$’ are the shape and scale parameters respectively. When ‘$\alpha$’ is approximated to the nearest integer, it tends to Erlang with density function

$$f_1(t) = [\alpha \cdot (\alpha - 1)!] (\lambda t)^{\alpha-1} / \alpha ! e^{-\lambda t}$$

The distribution function of Erlang model is given as:

$$F(t) = P(T \leq t) = 0^{\alpha-1} f_1(t) dt$$

$$F(t) = 1 - \sum_{i=0}^{n-1} (\lambda t)^i \exp(-\lambda t) / i! \; ; \alpha \text{ any positive integer}$$

Derivations of the various wave period parameters for zero up-crossing periods are presented elsewhere.

**Redefined significant wave period**

Significant wave period is redefined as the average period of the one-third highest zero up-crossing waves of a constant number of consecutive waves in a wave record. The Gamma model suggested for simulating zero up-crossing wave periods ($T_{rs}$) is modified by the method of characteristic function to describe the distribution of redefined significant wave period. The density function of Gamma model is:

$$df = [n^{\alpha/2} / (\beta^{\alpha/2} \Gamma(n))] \exp(-n(t/\beta)) / \beta^{n-1} dt$$

where $\beta = 1/\lambda$. Parametric relations derived from this model for predicting various redefined significant wave period parameters will have complicated expressions and hence modified Erlang distribution model is considered for redefined significant wave period modelling and prediction.

**Redefined significant wave period modelling and prediction**

The Erlang distribution model for redefined significant wave period is given as:

$$F(t) = 1 - \sum_{i=0}^{k-1} (\lambda t)^i \exp(-\lambda t) / i!$$

where $k$ is the number of one-third highest wave periods. The derivations of the various redefined significant wave period parameters are that of zero up-crossing wave periods and hence the derivation part is avoided in the following section.

**Mean redefined significant wave period ($T_{rs}$)**

The mean is

$$\bar{T}_{(rs)} = \alpha / (k \lambda)$$

and the variance is

$$\bar{T}_{(rs)^2} = \alpha / (k \lambda^2)$$

**Mean maximum redefined significant wave period ($T_{max(rs)}$)**

$$\bar{T}_{max(rs)} = \left[ 2 \alpha / \lambda - [1/(k \lambda (k \alpha - 1))] \times \sum_{i=0}^{k-1} (k \alpha + i) / (i ! 2^{k \alpha - i}) \right]$$

for sample size $n = 2$

**Average of the one-third highest of the redefined significant wave periods ($T_{rs(1/3)}$)**

$$p = \sum_{i=0}^{k-1} (k \lambda t)^i e^{-k \lambda t} / i! \; , \text{where} \; p = 1/3.$$

$$T_{rs(1/3)} = t + [1/(k \lambda)] \left[ \sum_{i=0}^{k-1} \sum_{j=0}^{i} (k \lambda t)^i / j! \left( \sum_{i=0}^{k-1} (k \lambda t)^i / i! \right) \right]$$

This is the expression for average of one-third highest of the redefined significant wave period. The value of ‘$t$’ in Eq. (7) is decided by Eq. (6).
Most frequent maximum redefined significant wave period ($T_{rMFM}$)
The most frequent maximum wave period of redefined significant wave period is given to be the root of the expressions
\[ \exp(-k\lambda t)\lambda^{k-2} = 0 \quad \ldots \quad (8) \]
or
\[ k\lambda t + \sum_{i=0}^{ka-1} \exp - (k\lambda t) [(\alpha + i - 1)(k\lambda t)^i - 2(k\lambda t)^{i+1}/i!] = k\alpha - 1 \quad \ldots \quad (9) \]
The Eq. (9) is found to give reliable results.

Analysis of extreme redefined significant wave periods ($R_p$)
The return period of an extreme redefined significant wave period is given by
\[ R_p = \left\{1 - \left[1 - \sum_{i=0}^{ka-1} (k\lambda t)^i \exp - (k\lambda t)/i!\right]^N\right\}^{-1} \quad \ldots \quad (10) \]
The extreme redefined significant wave period for a given return period $R_p$ is obtained as
\[ 1 - \sum_{i=0}^{ka-1} (k\lambda t)^i \exp - (k\lambda t)/i! = (1 - 1/R_p)^{1/N} \quad \ldots \quad (11) \]
The solution of Eqs. (10) and (11) is the $R_p$-t plane at $N=30$, where $N$-number of observations (here $10 \times 3 = 30$, assuming that there are 3 prominent redefined significant wave periods in a day).

Probability of realising a redefined significant wave period in a time $(m)$ less than the designated time $R_p$
\[ q = 1 - \left\{1 - \left[1 - \sum_{i=0}^{ka-1} (k\lambda t)^i \exp - (k\lambda t)/i!\right]^N\right\}^M \]
where $M = m/R_p \quad \ldots \quad (12)$
The zero up-crossing wave periods obtained from wave recorder charts at 0900hrs, 1200hrs and 1500hrs for January 1981 for Valiathura coast ($8°28'N$, $76°64'E$)(depth of recording, 5.5m)$^{12}$ were analysed by ‘zero up-crossing technique’. Twenty eight data sets were fitted to the Gamma distribution model and goodness of fit tested with $\chi^2$-test at 0.02 level of significance. The model fits in 79% cases confirming that the model has sufficient empirical support.

The frequency distribution of the redefined significant wave periods for a constant sample size of 6 [Average period of these 1/3 highest wave periods (ie.2)] were computed from zero up-crossing wave periods recorded at 0900 hrs, 1200 hrs, and 1500 hrs and were simulated using the modified Erlang distribution model suggested by the method of Characteristic Function$^{11}$. The $\alpha$ and $\lambda$ for 0900 hrs, 1200 hrs and 1500 hrs are respectively 25, 18, 15 and 1.7623, 1.2629, 1.1003. It is found that there is 100% fit to the theoretical model at 0.05 level of significance.

The redefined significant wave period is more realistic because the number ‘one-third highest waves (n)’ is known and it is a constant. It makes logical to modify the Erlang model to simulate the redefined significant wave period distribution. The parametric relations derived from this model provide more reliable estimations as evidenced from the following results.

The predicted mean maximum redefined significant wave periods ($T_{max(n)}$) for 0900 hrs, 1200 hrs and 1500 hrs are respectively 15.32 s, 15.59 s, 15.03 s whereas the computed mean redefined significant wave periods are 14.24 s, 13.83 s, 13.79 s. The computed and predicted average of the one-third highest of redefined significant wave periods ($T_{rs(n)}$) for 0900 hrs, 1200 hrs and 1500 hrs are 16.05 s, 16.53 s, 16.41 s and 16.41 s, 16.90 s, 16.41 s respectively. The RMS relative error is 0.018 and relative bias 0.149 indicating sufficiently accurate estimations. The mean maximum wave periods predicted are lower than the average of the one-third highest wave periods. This stems from the fact that the wave period parameters such as mean maximum, most frequent maximum and extreme wave periods of redefined significant wave periods are to be computed from the distribution of the maximum wave periods. The predicted most frequent maximum redefined significant wave periods for 0900 hrs, 1200 hrs and 1500 hrs are respectively 13.90 s, 13.86 s and 13.18 s.

Fig. 1 — Extreme redefined significant wave period (sec) distributions for given return periods (days)
The actual and predicted extreme redefined significant wave periods for a return period of 10 days for 0900 hrs, 1200 hrs and 1500 hrs are respectively 20.00 s, 22.00 s, 20.00 s and 20.19 s, 21.45 s and 21.29 s. The RMS relative error computed is 0.040 and relative bias is 0.015 showing that the predicted values are appreciably good. The distribution of extreme redefined significant wave periods follows an exponential curve, i.e., as the return periods anticipated become larger and larger, the wave period increases exponentially (Fig. 1). This seems to be more realistic than a uniform increase in wave periods as the return period increases uniformly.

The distribution of the probability percentage of realising an extreme redefined significant wave period (sec) in a time less than the designated return period (days)

By redefining the significant wave periods, the concept of significant wave approach has been made more effective as the number of waves is known in the new definition which is statistically more meaningful. Hence it is possible to suggest a modified Erlang model for the redefined significant wave period.

The authors are thankful to the Director, Centre for Earth Science Studies, Thiruvananthapuram, Kerala for providing the recorded wave data.

References

7. Anon, Wave statistics of the Arabian Sea, (Naval Physical and Oceanographic Laboratory, Cochin, India), 1978, pp. 204.