Performance of non-linear amplifier based phase locked loop systems in presence of channel perturbations

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The tracking performance of non-linear amplifier based conventional second order phase locked loop (PLL) and charge pump phase locked loop have been examined numerically by solving the system equations in the presence of lognormal type of fading signal. Some analytical results for non-linear amplifier based conventional phase locked loop and charge pump phase locked loop are also incorporated to confirm the simulation results and for the choice of optimum design system parameter.

Keywords: Non-linear amplifier, Phase locked loop, Signal

1 Introduction
The use of non-linear amplifier (NLA) in the conventional second order Phase locked loop (PLL) structure has been suggested in the literature. The inclusion of cubic type NLA in the conventional PLL structure enhances the transient response speed, increase the lock range as well as the hold-in-range of the PLL; but the cubic type NLA degrades the tracking performance of the said system when an additive noise or a co-channel tone interference signal accompanies the input signal. Again the presence of the amplitude and phase fading of the input signal due to time varying communication channel complicate the system operation considerably. In the context of mobile cellular communication as well as deep space communication, the studies of receiver system performance in the face of fading input signal are of practical importance. The channel perturbation problem occurs in these types of communication systems because in these networks the e.m. waves carrying information have to traverse long distance through different atmospheric layers to reach the ground based receiving antenna. In the first part of the paper the response of an NLA based conventional second order PLL to a lognormal type fading signal has been studied.

Next the study has been extended to explore the effects of lognormal fading input signal to the response of a PLL based on phase frequency detector (PFD), charge pump circuit (CP) and loop NLA. In recent years, CP-PLL circuits are extensively used as clock generators in variety of applications including microprocessors, wireless receivers and serial link transceivers etc. These systems have a number of merits over conventional PLLs, specifically CP-PLLs have wide acquisition and tracking ranges, very small steady state phase error and inherent frequency aiding operation. It is known that a PFD acts on the level transition instants of the input signal and the reference signal. The amplitude fading of the input signal affects the loop performance implicitly. The study includes the effect of signal fading in the tracking response of the system and the problem has been addressed analytically as well as numerically. The obtained results would help to choose optimum loop design parameters.

2 Formulation of the Problem and Approach of the Study
2.1 Mathematical Representation of the Input Signal
For lognormal fading channel the input signal to the PLL is considered as \( a(t)\sin(\omega t + \theta) \), where \( \omega \) and \( \theta \) are the carrier angular frequency and the input phase, respectively. Here \( a(t) \) is the amplitude of the input signal with lognormal probability density function (pdf) and defined as:

\[
a(t) = A_0 \exp(x(t))
\]

\[
A_0 \text{ being the amplitude of the transmitted signal entering the perturbing channel and } x(t) \text{ is a random...}
\]
process having normal distribution with mean \( m_x \) and power \( \sigma_x^2 \). \( \theta(t) \) is the phase perturbation with uniform distribution. The lognormal pdf of \( a(t) \) would be written as:

\[
P \left( \frac{a(t)}{A_0} \right) = \frac{1}{a(t) \sqrt{2\pi \sigma_x^2}} \exp \left\{ \frac{\ln \left( \frac{a(t)}{A_0} \right) - m_x^2}{2\sigma_x^2} \right\}
\]

\[
\text{... (2)}
\]

To characterize a lognormal channel usually parameters which are used, are the variance of the amplitude and phase perturbation processes. Moreover, the ratio of the bandwidth of the amplitude or phase perturbation process to the bandwidth of the phase locked loop receiver is also a parameter of practical importance. In actual cases, the bandwidth of the amplitude perturbation process is small compared to the PLL bandwidth. It can be added here that the phase perturbation and the amplitude perturbation processes have very weak correlation if not completely uncorrelated.

### 2.2 Mathematical Model of NLA

Figure 1(a and b) show the block diagram of the NLA based conventional PLL and the hardware structure of the NLA, respectively. The experimental input-output transfer characteristics are explained in the following way. In the limit of moderate input voltages, linear and cubic power terms of the input signal \( v_c \) are sufficient to express the output \( v_f \) of the NLA. But \( v_f \) would be limited to a prefix value \( v_k \) for \( v_c \) magnitude exceeding a value \( v_d \). Depending on the circuit components used to implement the NLA, one would have NLA design parameters \( K_d \) and \( K_a \). Thus, the input/output characteristics of the NLA is written as:

\[
v_f = K_d v_c + K_a v_c^3, \quad \text{for} \quad v_c \leq v_d
\]

\[
= \text{sgn}(v_c) \left| v_c \right|, \quad \text{for} \quad v_c \geq v_d
\]

\[
\text{... (3)}
\]

One can further define a parameter \( K_n \) given as \( K_n = K_a/K_d \) which may be varied to observe the effects of the NLA.

### 2.3 System Equation of Conventional PLL

Here the output of the loop voltage controlled oscillator (VCO) is considered as \( 2\cos(\omega t + \theta_0) \); where \( \omega \) and \( \theta_0 \) are the free running frequency and the estimated phase of the VCO. Then, one can write the phase governing equation of the NLA based conventional PLL as:

\[
\frac{d\phi}{dt} = \Omega - K_p F(s)v_f(t) + \frac{d\theta(t)}{dt}
\]

\[
\text{... (4)}
\]

where \( v_f(t) = a(t) K_p \sin \phi \)

\[
\text{... (5)}
\]

where \( K_p \) and \( K_c \) are the phase detector gain and the VCO sensitivity and \( \phi \) is the loop phase error given as:

\[
\phi = (\omega - \omega_i)t + (\theta - \theta_0)
\]

\[
\text{... (6)}
\]

where \( \Omega = (\omega - \omega_i) \) is the open loop frequency error and the filter transfer function is considered as:

\[
F(s) = \frac{1 + F_0 sT}{1 + sT}
\]

\[
\text{... (7)}
\]

where \( F_0 \) and \( T \) are the high frequency gain factor and the time constant of the loop filter. To examine the tracking performance of the loop Eqs (3), (4), (5) and (7) are numerically solved in real time.

![Fig. 1(a)—Block diagram of the NLA based conventional PLL](image)

![Fig. 1(b)—Hardware structure of the NLA used in the PLL](image)
2.4 System Equation of the CP-PLL

Figure 1(c) shows the block diagram of non-linear amplifier based CP-PLL system. For a CP-PLL having edge sensitive PFD, the amplitude of the lognormal fading signal would be limited and the angle part of the signal would be actual concern. So the presence of lognormal fading signal can be taken into account by adding a random phase component ($\theta(t)$) with the input signal phase. Following the same mathematical analysis of Gardner\textsuperscript{9}, one can define tracking phase error ($\phi_e$) for this system as the difference of input signal phase and the VCO phase. Mathematically, one can write the tracking phase error at the $N+1$ th active transition instant in terms of $N$ th active transition instant of the NLA based CP-PLL system as:

$$D = \left( \frac{K_v K_a C_1}{r_i^2} \right) \left( \frac{2\pi K}{K_v} \right) \left( \frac{2\pi K}{K_v} \right) \left( \frac{1+\frac{2\pi}{r_i}}{r_i} \right) - 2v_x(N)$$

where the parameters $K (= K/I/R/2\pi)$, $x (= K/\omega)$, $K_v$, $R$, $\Delta\omega$ are the normalized loop gain, normalized loop bandwidth, VCO sensitivity, resistance used in the loop filter and frequency error between noise free input signal and VCO output, respectively. Here $C_1=2\pi x$ and $r_i=\omega R_C=\omega r$. Also $\omega$ and $I_P$ are the input signal frequency and the pump current magnitude, respectively.

3 Simulation Results on Tracking Performance

3.1 Conventional PLL With NLA

In this numerical simulation the lognormal type of fading signal is applied when the loop is in locked condition. For numerical simulation, slow amplitude fading of input signal is considered only by putting $d\theta(t)/dt=0$. Here the magnitude of normalized frequency detuning $\Omega/(A_0 K_a K_f K_r)$, high frequency gain ($F_0$) and normalized filter time constant $T_f/(A_0 K_a K_f K_r)$ is considered as 0.5, 0.02 and 1.0. Numerical simulation results are presented in Figure 2 (a and b). Figure 2(a) shows the variation of the mean square (MS) value of normalized output phase tracking error with the input signal variance for different magnitude of non-linear parameter of the NLA. Here normalization is made with respect to linear or conventional system. From Figure 2(a), it is
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It is very much difficult to get a closed loop analytical solution of output phase error of NLA based conventional PLL system. So to make the analysis tractable the amplifier non-linearity is replaced by suitable sine-type non-linearity with the help of a describing function method\(^{11}\). Consider that the non-linear system under study has an input/output characteristic given as:

\[ v_f = f(v_c) \quad \ldots (9) \]

where \(f(.)\) denotes a non-linear function. Now for a situation when \(v_c\) is a sinusoidal signal, \(v_f\) can be expressed as a sum of two components, one being \(v_c\) multiplied by a gain factor \((A_f)\) and the other being distortion signals which are higher harmonic of \(v_c\). If

3.2 NLA Based CP-PLL System

To study the tracking performances of the NLA based CP-PLL system Eqs 8 (a and b) are numerically solved. In this numerical simulation, the magnitude of \(K/K_v\) and normalized loop damping parameter is considered as 0.1 and 0.707, respectively. These simulation results are shown in Fig. 3. Figure 3 shows the variation of MS tracking error with the input signal variance for different magnitude of non-linear parameter and the magnitude of output MS tracking error decreases with non-linear parameter for a particular magnitude of input signal variance. So, the inclusion of NLA in the conventional CP-PLL structure enhances the tracking performance of the CP-PLL system in the presence of lognormal fading signal. So, one can conclude that the NLA based CP-PLL system is more suitable than the conventional CP-PLL system in the face of fading input signal.

4 Analytical Studies on Tracking Performance

4.1 Conventional PLL Using NLA in the Face of Fading Signal

It is very much difficult to get a closed loop analytical solution of output phase error of NLA based conventional PLL system. So to make the analysis tractable the amplifier non-linearity is replaced by suitable sine-type non-linearity with the help of a describing function method\(^{11}\). Consider that the non-linear system under study has an input/output characteristic given as:

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where \(f(.)\) denotes a non-linear function. Now for a situation when \(v_c\) is a sinusoidal signal, \(v_f\) can be expressed as a sum of two components, one being \(v_c\) multiplied by a gain factor \((A_f)\) and the other being distortion signals which are higher harmonic of \(v_c\). If
the output of the non-linear network is passed through a proper low pass filter, distortion terms can be ignored and $v_f$ can be written as:

$$v_f = A_g v_e$$

(10)

where the quantity $A_g$ is the describing function for the non-linearity. Using the technique outlined above, one can write the output of the NLA as $A_g(t) \sin \phi$, where $A_g(t)$ is given by:

$$A_g(t) = \frac{4v_e}{\pi} \cos \beta + \left( a(t) K_p K_g \right) \frac{\pi}{\pi}$$

$$\times \left[ 2 \left( 1 + \frac{3}{4} K_n a(t)^2 K_p^2 \right) - \left( 1 + K_n a(t)^2 K_p^2 \right) \sin 2\beta \right]$$

$$+ \frac{K_n a(t)^2 K_p^2}{8\pi} \sin 4\beta$$

(11)

where $\beta$ are equal to $\arcsin (v_e / A_0)$. For analytical solution, one must consider the time variations of $a(t)$ and $\theta(t)$ separately. This is logical in the context of practical situations also since $a(t)$ and $\theta(t)$ process are uncorrelated. To make the analysis tractable, it is considered that the phase perturbation process is very slow compared to the amplitude fluctuation. In this situation, one considers $\theta(t)$ to be time invariant and put $\frac{d\theta(t)}{dt} = 0$. The amplitude of the input signal varies according to lognormal fading law. This is the case of slow amplitude fading. Now if one can consider some additive Gaussian noise is present with the fading signal, then using the same technique as in Ref. 12, one can write the output phase error variance ($\sigma^2_\phi$) after a bit of algebra as:

$$\sigma^2_\phi = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n I_n(\alpha)}{n^2 I_0(\alpha)}$$

(12)

where $B$ is the bandwidth of the system pre-filter. Here loop CNR has been defined in terms of fixed amplitude $\sqrt{2}A_0$. If the channel be non-fading one and channel efficiency be unity, $A_0^2$ would be power transmitted by the transmitter. From Eq. (12), it is clear that the output phase error variance would be related with the instantaneous values of the signal amplitude. Then averaging the instantaneous variance, one can obtain the average variance of phase tracking error as:

$$\langle \sigma^2_\phi \rangle \approx \int_0^\infty \sigma^2_\phi P\left( \frac{a}{A_0} \right) d\left( \frac{a}{A_0} \right)$$

(13)

For a given loop CNR and input signal variance of fading signal the average MS value of tracking error are obtained from Eq. (13) for different magnitude of non-linear parameters. These analytical results are also potted in Fig. 2(a). From these analytical results, it is seen that the magnitude of MS tracking error is greater for NLA based system than the conventional system. It indicates that the performance of NLA based system is degraded due to channel perturbation. Thus, the numerical simulation and approximate analysis give qualitatively equivalent results. From Fig. 2(a) it is also clear that there is a good agreement between the analytical and numerical simulation results. But within a limited zone of input signal variance, analytical results differ from numerical simulation results. This disparity between analytical and numerical simulation results is mainly due to the approximation made to derive the analytical expression of output phase error variance.

### 4.2 CP-PLL System using NLA in the Face of Fading Signal and Choice of Optimum Design Parameter

Like NLA based conventional PLL, it is also very much difficult to get an analytical solution of tracking phase error for NLA based CP-PLL system. One can get only an approximated solution of the said system by neglecting the higher order non-linear terms of Eqs 8 (a and b). Then, one can write the discrete linear difference equation of the loop phase error in the face of lognormal fading input signal with the help of Eqs 8 (a and b) as:

$$\phi_e(N+1) = a_1\phi_e(N) + b_1\phi_e(N-1) + \theta(N+1) - \theta(N)$$

(14)

where
Here the sample values of \( \theta(t) \) are mutually statistically independent and because of obvious reasons, \( \theta(t) \) sample of present instant is independent of \( \phi_e \) values of previous instant. Considering \( \theta(t) \) as a zero-mean process, the mean values of \( \phi_e \) is obtained as zero from Eq. (14). The simplified expression for the MS value of tracking error \( (\phi_e) \) can be obtained as:

\[
\phi_e^2 = \frac{2\bar{\theta}^2}{(1+b)(1+a_i-b)}
\]  

(15)

If the input random phase \( \theta(t) \) is just like a band limited Gaussian noise in nature, then the pdf of \( \theta(t) \) is known\(^1\) and using this pdf one can find the MS value of \( \theta(t) \) in terms of the input signal to noise power ratio (SNR), \( \rho \) as follows:

\[
\bar{\theta}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta^2 \left( \exp(-\rho) + \sqrt{\pi \rho} \cos \theta \exp(-\rho \sin^2 \theta) \right) \times \left( 1 + \text{erf} \left( \sqrt{\rho} \cos \theta \right) \right) d\theta
\]  

(16)

where \( \text{erf}(x) \) is the error function of argument \( x \). For a given SNR, the MS value of tracking error obtained from Eqs (15) and (16) is plotted in Figure 3 with the simulation results as a function of non-linear parameter of the NLA. To get Eq. (15), one can consider some approximation by neglecting higher order non-linear terms. For these reason, from Figure 3 it is seen that the analytical results do not correctly follow the simulation results. So, this analytical method is just like an approximate quasi-linear approach. But one can get the same conclusion from analytical and simulation results.

Eq. (15) can also be used to find the stability condition of the system or to choose the optimum design parameter for the stable loop operation. Since MS value of tracking phase error cannot be negative, for a stable system one must have:

\[
K_s < 2 \left( \frac{\omega \tau}{\pi KK_s \tau} - \left( 1 + \frac{\pi}{\omega \tau} \right) \right) \left( \frac{K_s}{2\pi K} \right)^2
\]  

(17)

For conventional CP_PLL system, this equation reduces to:

\[
K_s \tau < \left( \frac{\pi}{\omega \tau} \right) \left( 1 + \frac{\pi}{\omega \tau} \right)^{-1}
\]  

(18)

Incidentally, this obtained condition is the same as that obtained in Ref. 9 through a different approach.

5 Conclusion

The present paper reports the tracking performance of the NLA based conventional second order PLL and CP_PLL in the presence of lognormal fading input signal. The results of numerical simulation for conventional second order PLL show that the tracking performance of the said system is degraded in the presence of fading environment. The degradation of system performance is more for higher magnitude of non-linear parameter of the NLA and for higher magnitude of input signal variance. It is seen that the MS tracking error attains the maximum value at a particular magnitude of non-linear parameter of the NLA. It can also be seen that the maximum magnitude of the applied input signal variance depends on the non-linear parameter of the NLA. This observation also confirms the degradation of system performance in the presence of fading environment. In this paper, the tracking performance of NLA based CP_PLL system in the face of fading signal is examined by analytical means and simulation studies. It is observed that the NLA based CP_PLL system shows better tracking performance than the conventional one in presence of fading signal. Finally, the stability limit of the NLA based CP_PLL system is derived analytically. This expression is very much important for the proper choice of optimum design parameter.

References