Collapse behaviour and simplified modeling of triangular cross-section columns

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Received 5 September 2008; accepted 18 February 2009

This paper investigates and develops the collapse behaviours and characteristics of thin-walled columns with triangular cross-section under axial compression and pure bending. Two bending modes, inward bending and outward bending are discussed. The axial and bending resistances of the triangular columns are formulated with mathematical equations of simple forms. Simplified finite element models for the triangular columns are then developed based on the derived equations to simulate their axial buckling and bending collapse behaviours during crashes. The developed simplified models consist of beam elements and spring elements. Numerical results and comparisons show that the developed simplified models can replace detailed models in using for crashworthiness analysis and save a lot of computing time and modeling efforts. The presented simplified models and modeling efforts can be extensively applied in thin-walled structural analysis and design.

Keywords: Axial buckling, Bending collapse, Thin-walled column, Triangular cross-section, Simplified model, Crashworthiness analysis

As an important energy-absorbing component, thin-walled column of triangular cross-section has aroused a lot of research interests. Researchers have thoroughly studied the crushing behaviour of the thin-walled triangular column during crashes which subjects to axial loading and bending moment. Krolak et al.\textsuperscript{1} conducted experiments to analyze axial collapse behaviour and load caring capacity of the triangular cross-section columns with single-cell cross-section as well as multi-cell one. Ye et al.\textsuperscript{2} performed an analytical analysis to predict the force-shortening relationship of the triangular column based on the real plastic mechanism. Through their study, simple formulae were derived to express the force-shortening curve of such column. Theoretical study of the collapse of the thin-walled triangular column is also presented by Jakubowski\textsuperscript{3}. Furthermore, in his study, such column subjects to compound loading instead of the axial load only. Besides above literatures, Abramowicz and Wierzbicki\textsuperscript{4} developed analytical models to predict the axial crushing behaviours of thin-walled multi-corner columns, which can also be applied to the research of the axial crushing of the triangular column.

In addition to axial collapse, Kotelko and Korlak\textsuperscript{5} studied the collapse mechanisms of the triangular cross-section columns under pure bending, based on which the moment-rotation relationship of the triangular columns are formulated and validated through comparing to experimental data.

One objective of this paper is to develop simplified finite element (FE) model for the thin-walled triangular column by employing the previously developed collapse theories. The developed simplified model will then be used for numerical crashworthiness analysis. As concluded in previous literatures\textsuperscript{6-11}, compared to detailed FE model, the application of the simplified model can save a great amount of modeling efforts and computing time when being used for crashworthiness analysis. Practically, a thin-walled column absorbs the most impact energy during crashes. Therefore the simplified model must be capable of capturing the axial buckling and bending characteristics of the thin-walled column. Liu and Day\textsuperscript{6-10} systematically developed methodologies of generating simplified models for both straight and curved regular thin-walled columns, which can properly reflect the crushing behaviours caused by axial loading and pure bending. The modeling approach presented by Liu and Day is followed here to develop simplified models for the triangular column.

The developed simplified models are used for crashworthiness analysis and validated by comparing the results to those from the detailed models. It is proved that the simplified triangular column models presented in this paper can well replace the detailed models and be used for crashworthiness analysis. In this study, explicit FE code LS-DYNA is applied in creating FE models and running the numerical analyses.

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Under Axial Compression

In this section, the collapse behaviours of the triangular columns subjected to axial compression are firstly analyzed. Mathematical models are formulated to evaluate the axial resistance of such columns based on previous findings and a simplified model is created to simulate the axial buckling behaviour of the triangular columns.

Collapse behaviour

The method developed by Abramowicz and Wierzbicki\(^4\) is applied to formulate the axial resistance of the thin-walled triangular columns. According to the authors, the axial crushing resistance for a multicorner thin-walled tube equals to the appropriate crushing force per one element times the number of corner elements, \(n\). The mean crushing force per one element is

\[
\frac{P_m}{M_0} = \left( A_1 \frac{r}{t} + A_2 \frac{a}{H} + A_3 \frac{H}{r} \right) \frac{2H}{\delta_{ef}} \quad \cdots (1)
\]

and can be rewritten in the form:

\[
\frac{P_m}{M_0} = 3(A_1 A_2 A_3)^{1/3} (a/t)^{1/3} \frac{2H}{\delta_{ef}} \quad \cdots (2)
\]

where

\[
M_0 = (\sigma_0 t^2) / 4 \quad \cdots (3)
\]

where \(P_m\) is the mean crushing force; \(M_0\) is fully plastic moment per unit length of section wall; \(A_1\) through \(A_3\) are coefficients and from (ref. 4) \(A_1 = 4.44, A_2 = \pi\) and \(A_3 = 2.30\), respectively; \(a\) is edge length of octagonal cross section; \(t\) is wall thickness; \(2H\) is length of one plastic fold generated when the tube buckled (see Fig. 1), \(\delta_{ef}\) is called “effective crushing distance” and \(\delta_{ef}/2H\) was measured as 0.73 from earlier experimental observations\(^4\), \(\sigma_0\) is the material’s energy equivalent flow stress that approximately equals to: \(\sigma_0 = 0.92\sigma_u\)

Substitutes all the values \((A_1, A_2, A_3, \text{ and } \delta_{ef})\) into Eq. (2) and multiplies it by three for the triangular section, the axial resistance of the thin-walled triangular columns can be approximated by the equation:

\[
\frac{P_m}{M_0} = 39.17(a/t)^{0.33} \quad \cdots (4)
\]

Substitutes Eq. (3) into Eq. (4), the mean crushing force can be directly calculated based on the given octagonal tube’s dimensions and material properties as:

\[
P_m = 9.8\sigma_0 t^{1.67} a^{0.33} \quad \cdots (5)
\]

To find the length of plastic fold \(2H\), minimizing Eq. (1) with respect to \(H\) by using \(\partial P_m/\partial H=0\) yields

\[
H = 0.97\sqrt{ta^2} \quad \cdots (6)
\]

The calculated \(P_m\) is the axial resistance of the triangular columns and Eqs (5) and (6) will be used to determine the characteristics of the spring elements used in the simplified models.

Detailed FE model

The detailed model for the thin-walled triangular cross-section column is first created and shown in Fig. 2, which is a straight column and will subject to the axial loading when impact occurs. The detailed

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Fig. 1 — Length of half plastic folding wave, \(H\)

Fig. 2 — Detailed thin-walled triangular cross-section column model (straight)
model is modeled using full integration shell element: 4-node Belytschko-Tsay shell element with 5 integration points through the thickness. During the crashworthiness analysis, the column model impacts a rigid wall at an initial velocity 15 m/s. The length of the column model is 300 mm, the side width of the triangular cross-section $a$ is 40 mm, and the wall thickness $t$ is 1.5 mm. In this study, the column model is made of mild steel and the plastic kinematic material (material type 3) is used to model the material behaviour in LS-DYNA. Detailed material properties for the mild steel are listed in Table 1.

The detailed FE model is used for the numerical crashworthiness analysis and the dynamic results are compared to those yielded from the simplified model.

**Simplified FE model**

As mentioned before, the simplified model consists of beam elements and spring elements. Beam elements can be easily created by assigning proper cross-sectional information onto them. Eqs (5) and (6) are employed to determine the force-shortening characteristic of the spring elements. Substituting the material properties listed in Table 1 into Eqs (5) and (6), the length of one plastic fold is calculated as $2H = 26$ mm, and the mean crushing force $P_m$ is 26.9 kN. Thus, the spring element is fully determined, which simulates the axial buckling behaviour of the triangular columns, and its $F$–$\delta$ relationship is plotted in Fig. 3. With this characteristic, the spring begins to deform when the crushing force reaches 26.9 kN. After its deformation reaches 226 mm, the spring fails and stops deforming.

The beam elements used in the simplified model should have triangular cross-section with proper dimensions. LS-DYNA does not directly provide the triangular section type, however, it offers the trapezoidal section type (SECTION-07) and the triangular section can be derived from the trapezoidal section by setting one base equals 0.

The simplified FE model (Fig. 4) for the thin-walled triangular column is then created using Liu and Day’s method. In the developed simplified model, the Hughes-Liu beam elements are used for building a pure beam-element model with triangular cross-sectional information assigned. Afterwards, the entire beam is divided into several equal segments with the length of 26 mm, which is the length of one plastic fold ($2H$). These segments are connected by the developed spring elements, whose force-displacement relationship is displayed in Fig. 3. Appropriate boundary conditions are also applied in order to ensure that the simplified model only deforms axially.

**Analysis and comparisons**

In order to validate the developed simplified model, the simplified model (Fig. 4) was used for the same crashworthiness analysis as the detailed model (Fig. 2). Important results were recorded from the analyses and compared in Table 2 as well as plotted in Figs 5-7.

From Table 2, it can be seen that the results yielded from the simplified model reaches a good agreement to those from the detailed model. All the errors are below 10% and the simplified model consists of much less and simpler elements. Hence, the developed simplified straight triangular column model is

<table>
<thead>
<tr>
<th>Table 1 — Mild steel properties</th>
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<tbody>
<tr>
<td>Material type</td>
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<tr>
<td>Young’s modulus</td>
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<tr>
<td>Density</td>
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<td>Yield stress</td>
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<td>Ultimate stress</td>
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<td>Hardening modulus</td>
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<td>Poisson’s ratio</td>
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qualified for replacing the detailed model to be used for crash analyses. The presented simplified modeling method as well as the developed simplified model is validated.

**Under Pure Bending**

The collapse behaviours of the triangular columns subjected to pure bending and the bending resistance of such columns presented by Kotelko and Krolak\(^5\) is applied to develop simple equations that can properly represent the bending collapse of the triangular columns. Such equations are applied to develop the simplified curved triangular column model.

**Bending behaviour**

**Inward bending**

Two types of bending behaviours: inward bending and outward bending are depicted in this part and simple equations are derived to predict the respective bending resistance. Figure 8 plots the inward bending collapse mechanism, which was used by Kotelko et al.\(^5\) to formulate the inward bending resistance of the triangular columns. It is well known that under

| Detailed model | Simplified model | Difference (%)
<table>
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<tr>
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<tbody>
<tr>
<td>Peak crushing force (kN)</td>
<td>43.6</td>
<td>40.0</td>
</tr>
<tr>
<td>Absorbed energy (kJ)</td>
<td>2.67</td>
<td>2.83</td>
</tr>
<tr>
<td>Global displacement (mm)</td>
<td>145.5</td>
<td>147.2</td>
</tr>
<tr>
<td>Nodes/Elements</td>
<td>1224/1200</td>
<td>110/108</td>
</tr>
</tbody>
</table>

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**Table 2 — Analysis results from detailed and simplified straight triangular column models**

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**Fig. 5** — Crushing forces on straight triangular section column models

**Fig. 6** — Energies absorbed by straight triangular section column models

**Fig. 7** — Global displacements of straight triangular section column models

**Fig. 8** — Inward bending collapse mechanisms of thin-walled triangular column\(^5\)
pure bending, all the impact energy is absorbed by the plastic hinges\textsuperscript{13-14}. From Fig. 8, the energy absorbed during rotation of the global plastic hinge is considered to dissipate along seven types of hinge lines (line 1-7). Therefore, the total absorbed energy $E(\theta)$:

$$E(\theta) = 2E_1 + E_2 + 2E_3 + 4E_4 + 4E_5 + 4E_6 + 2E_7 \quad ... (7)$$

where $\theta$ is the rotation angle.

For simplification, in this study the triangular sectional profile is an equilateral triangle with side length $a$ and wall thickness $t$ – apparently, the bending behaviour of triangular columns with arbitrary sectional profiles can be derived following the same approach.

According to Kotelko and Krolak\textsuperscript{5}, the energies dissipated along hinge lines 1-3 are:

$$E_1 = M_0 a (\alpha / 2 - \theta / 2) \quad ... (8)$$

$$E_2 = M_0 a \alpha \quad ... (9)$$

and

$$E_3 = M_0 l_3 \alpha \quad ... (10)$$

where $\alpha$ is shown in Fig. 8 and according to Kotelko and Krolak\textsuperscript{5}

$$\alpha = 2 \arccos \left(1 - \frac{2a}{H} \sin \frac{\pi}{3} \cos \frac{\theta}{2} \right) \quad ... (11)$$

$l_3$ is the length of hinge line 3, which can be determined from Figs 8 and 9 as:

$$l_3 = \frac{-a + \sqrt{a^2 + 4\sqrt{3}aH \sin(\alpha / 2 - \theta / 2)}}{2} \quad ... (12)$$

The energy absorbed by hinge line 4 is

$$E_4 = M_0 H \frac{\pi}{3} \quad ... (13)$$

The energy absorbed by line 5 is rationally simplified from Kotelko's conclusion in order to make it more fit for simplified modeling. As discussed in one of Liu's literatures\textsuperscript{9}, the energy absorbed by such plastic mechanism can be estimated as

$$E_5 = M_0 \frac{H^2 + a^2}{2} \frac{H}{a} \quad ... (14)$$

The energies absorbed by line 6 and 7 are approximated by Kotelko and Krolak\textsuperscript{5} as

$$E_6 = M_0 \frac{l_3}{0.07 - \theta / 70} \quad ... (15)$$

And

$$E_7 = \frac{4}{3} M_0 \frac{l_3 (a - l_3)}{0.07 - \theta / 70} \quad ... (16)$$

Substituting Eq. (8-16) into Eq. (7), we can have the overall amount of absorbed energy. The inward bending resistance that is essentially the moment-rotation relationship then can be determined using

$$M(\theta) = \frac{dE(\theta)}{d\theta} \quad ... (17)$$

After taking several proper assumptions, the $M(\theta)$-$\theta$ relationship is represented as

$$M(\theta) = \left[ M_0 a \left( \sqrt{\frac{a}{H\theta}} - 1 \right) + l_3 \sqrt{\frac{a}{H\theta}} + 2 \sqrt{\frac{aH \theta}{3}} \right]$$

$$+ \frac{(H^2 + a^2)}{\sqrt{aH \theta}} + 4 \frac{l_3 (0.07 - \theta / 70) + l_3 / 70}{(0.07 - \theta / 70)^2}$$

$$+ \frac{8 (a l_3 - 2 l_3)(0.07 - \theta / 70) + l_3 (a - l_3) / 70}{(0.07 - \theta / 70)^2} \quad ... (18)$$

where

$$l_3 = \sqrt{\frac{\sqrt{3}}{2}} \frac{a}{a \sqrt{aH \theta} - aH \theta}{4} \quad ... (19)$$

Fig. 9 — Geometry of the deformed triangular cross section
Meanwhile, in the real impact test, the triangular column stops bending because of jamming. As explained by Kotelko and Korlak\textsuperscript{5}, jamming takes place when the angle $\theta$ reaches its limit value:

$$\theta_j = 2\arctg\left(\frac{c}{2a\sin(\pi/3)}\right)$$

... (21)

These equations will be used to define the characteristic of the rotational spring when used for simulating the inward bending of the thin-walled triangular column model.

**Outward bending**

Collapse mechanism of outward bending is plotted in Fig. 10a and verified through a computer simulation Fig. 10b. Comparing Fig. 10 to Fig. 8, it is found that for the plastic mechanism of the outward bending is almost the same as the one of the inward bending. The only difference is that during the inward bending, the plastic hinge (hinge lines 5-7) moves outside while during the outward bending, these hinge lines move inside. Thus, the same equations (Eqs (18)-(20)) can also be used for depicting the outward bending behaviour of the thin-walled triangular columns.

**Detailed FE model**

Using the same shell elements introduced before, a detailed FE model for curved thin-walled triangular column is created and shown in Fig. 11. This is a curved triangular column model that will perform bending collapse during the crashworthiness analysis. As concluded by Wierzbicki \textit{et al.}\textsuperscript{13,14}, during the impact of such curved thin-walled columns, the deformation is concentrated on the global plastic hinges; therefore most impact energy is absorbed there. Meanwhile, the axial deformation at those plastic hinges is very small compared to the rotations, thus, such crushing behaviour of the curved thin-walled column model can be considered as pure bending. The column is made of mild steel and the material properties are the same as listed in Table 1. Similarly, during the crashworthiness analysis, the curved column longitudinally impacts the rigid wall at 15 m/s, the side width $a$ is 60 mm and its wall thickness is 1.5 mm.

**Simplified FE model**

The simplified curved triangular column model (Fig. 11) then can be developed following Liu and Day’ modeling method\textsuperscript{6,8-10}. In the simplified model, the straight beam segments are modeled using Hughes-Liu beam elements, and the developed nonlinear rotational spring elements are used to model the local plastic hinges. Eq. (18-21) are used to define the bending resistance of such nonlinear rotational spring element (Fig. 12). From Fig. 12, it can be seen that under the bending moment, the rotational spring rotates with the displayed moment-rotation characteristics and stops bending when the angle $\theta$ reaches the jamming angle $\theta_j$ [see Eq. (21)]. The plotted moment-rotation relationship was simplified from Kotelko and Krolak’s work by adopting reasonable assumptions, the $M(\theta)$-$\theta$ curve presented in Fig. 12 is very close to the one sketched\textsuperscript{5}. Similarly, appropriate boundary conditions are applied in order to guarantee that the simplified model only bends along X-Y plane. As shown in Fig. 13, the
created simplified model for the curved triangular column can be considered as three straight segments which are connected by three plastic hinges. From earlier studies\textsuperscript{5,9}, this is a reasonable approximation of the real curved column model on the basis of crashworthiness criteria.

**Analysis and comparisons**

After the completion of the modeling, the simplified model undergoes the same crash analysis that has been performed on the detailed model. The analysis results from both models are compared. Table 3 and Figs 14-16 display all the results from the comparison.

From the results of the comparisons, it is verified that the simplified model created can generate good results while saving much more modeling efforts and computing time because of its simple configuration. Table 3 shows that errors of the analysis results are within 6%. The quality of the simplified curved triangular column model is therefore confirmed.
In this paper, the crash behaviours of thin-walled triangular section columns are investigated, including the straight and the curved column. The bending resistance formulated by Kotelko et al. is simplified and another bending mode, outward bending, is also discussed for the purpose of simplified modeling. On the basis of current collapse theories, the simplified models for such columns are developed. Respective modeling methods are also summarized and presented. In these simplified models, beam elements and spring elements are employed to simplify the existing detailed models by replacing the shell elements. The validity of these simplified models is verified through a series of crashworthiness analyses and comparisons. It is also verified that the derived axial resistance and bending resistance of the thin-walled triangular columns are valid for estimating the triangular columns’ crushing behaviours, and are usable for determining the spring elements during the simplified modeling.

The comparison of various detailed models shows that the simplified models with reduced size and elements can save many more modeling labours and computing time. Such advantages make the simplified models most suitable for computer simulation and product design. For an example, following the approach put forward by Liu, the simplified models and modeling methods presented in this paper can be applied in optimum design of the thin-walled triangular columns.

Besides the advantages, the developed simplified models also have their limitations. As demonstrated here, the simplified model can accurately capture the important characteristics during the crash analyses. However, due to its concept structure, it cannot faithfully represent different collapse modes of the thin-walled triangular column under different impact conditions. As observed in earlier studies, the detailed thin-walled triangular columns could show two distinct collapse modes in axial collapse and four bending modes in bending collapse, which gave researchers more conveniences in revealing the essence of the thin-walled columns’ crushing. Thus, the detailed models are valuable to minutely investigate the collapse models and features, while the simplified models are suitable for approximately simulating the collapse procedure and to roughly predict the crushing behaviours of thin-walled triangular columns with different geometries.

Reference