Convection due to oblique magnetic field in the penumbral region of sunspot

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The linear stability of convection due to oblique magnetic field in the penumbral region of sunspot has been investigated. We have obtained the values of Takens-Bogdanov bifurcation points and co-dimension two-bifurcation points by plotting graphs of neutral curves corresponding to stationary and oscillatory convection for different values of physical parameters relevant to convection in the penumbral region of sunspot.

Keywords: Convection, Penumbra, Bifurcation points

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1 Introduction

The interaction between thermal convection and magnetic field has received considerable attention because of its importance in geophysical and astrophysical problems. The original motivation for studying magneto-convection arose from the attempts to explain the origin of sunspots and starspots. Sunspots and star spots are dark, because thermal convection is suppressed by a strong magnetic field, which allows only a weaker time dependent motion. The magnetic field in sunspots is vertical in umbral region (typically 3000 G) but spreads outward to nearly horizontal in the penumbral region (typically 1000 G) \cite{Durrant1}. The angle $\phi$ which magnetic field in sunspot makes with the vertical axis is given by Bray and Loughhead\textsuperscript{2} as

$$\phi = \frac{r}{b} \ 75^\circ$$

where $r$ is the distance from the center of the sunspot and $b$ the radius of the sunspot. Thus, the field is vertical at the center of sunspot, but as one moves away from the center, it becomes more and more inclined to the vertical, reaching at an angle of $75^\circ$ to the vertical at the boundary of the penumbra\textsuperscript{3}. Tagare and Murali\textsuperscript{4} considered the problem of magneto-convection due to horizontal magnetic field in the penumbral region. They have used Boussinesq approximation. Busse\textsuperscript{5} has considered the problem of magneto-convection in an inclined magnetic field. In a sunspot penumbra, $\kappa/\eta \approx 10^3$ at the photosphere, where the magnetic diffusivity, $\eta$ is much smaller than the thermal diffusivity, $\kappa$ and we expect to find oscillatory convection appearing through the double diffusive effects, which introduce phase differences between thermal and magnetic perturbations.

As super-adiabatic gradient is increased, oscillatory convection eventually gives way to steady stationary convection. The opacity increases rapidly with depth, due to ionization of H and He, leading to a sharp reduction in the thermal diffusivity, $\kappa$. Following Meyer et al.\textsuperscript{6}, we can distinguish three layers. In the topmost layer (0 km $\leq z \leq 500$ km), $\kappa/\eta > 1$, then comes a region (500 km $\leq z \leq 2000$ km), where $\kappa/\eta > 1$. If Chandrasekhar number $Q$ is greater than critical Chandrasekhar number $Q_c$ and $\kappa/\eta > 1$, we get Hopf bifurcation point. If $Q$ is less than $Q_c$ then we get pitchfork bifurcation point. At $Q = Q_c$ and $\kappa/\eta < 1$, we get secondary bifurcation point known as Takens-Bogdanov bifurcation point, where neutral curves of stationary and oscillatory convection meet and the frequency on the neutral curve of oscillatory convection approaches zero. Co-dimension two-bifurcation point is also a point of secondary bifurcation, where Rayleigh number for the onset of oscillatory convection coincides with Rayleigh number for the onset of stationary convection (but at different wave numbers).

Finally, we have a region (2000 km $\leq z \leq 20000$ km) where $\kappa/\eta < 1$ and we get only stationary convection,
which is likely to become time-dependent in a highly nonlinear regime. In this paper, we study the magneto-convection in a deep inner penumbra region with a magnetic field inclined as much as 75° to the vertical. In Sec 2, we describe our model of solar penumbral region and write basic equations of the problem of magneto-convection in the solar penumbral. In Sec 3, we study the linear stability analysis of the problem. In Sec 4, we write conclusions of our study.

2 Basic equations

Let us consider an electrically and thermally conducting fluid between two horizontal planes with adverse temperature gradients. We assume that an externally impressed magnetic field \( H = H_o (\sin \phi, 0, \cos \phi) \) lies in the \( \text{xz} \)-plane and is inclined at an angle \( \phi \) to the direction of the vertical. Acceleration due to gravity is \( g = g(0, 0, 1) \). Let \( d \) be the depth of the solar convection zone in the penumbral region of the sunspot (or the distance between two horizontal planes). Let \( \rho_o \) be the density, \( v \) the viscosity, \( \eta \) the magnetic diffusivity (electric resistivity) of electrically and thermally conducting fluid in the solar convection zone, \( \mu_m \) a magnetic permeability, \( \beta \), an adverse temperature gradient, \( \alpha \) and \( \kappa \) respectively coefficients of thermal expansion and thermal diffusivity of the fluid.

The dimensional equations for magneto-convection under the Boussinesq approximation are [Chandrasekhar, Bekki and Karakosavia]

\[
\begin{align*}
\nabla \cdot V' &= 0, \quad \nabla \cdot H' = 0 \quad \ldots \quad (1) \\
\left[ \frac{\partial V'}{\partial t'} + (V' \nabla)V' \right] &= \\
- \frac{\nabla P'}{\rho'_o} + \frac{4\pi\mu_m}{\rho'_o} (V' \times H') \times H' \quad \ldots \quad (2) \\
\frac{\partial T'}{\partial t'} + (V' \nabla)T' &= k\nabla^2 T' \quad \ldots \quad (3) \\
\frac{\partial H'}{\partial t'} &= V' \times (V' \times H') + \eta \nabla^2 V' \quad \ldots \quad (4)
\end{align*}
\]

We use a Cartesian system of co-ordinates whose dimensionless vertical co-ordinate \( z' \) and horizontal co-ordinates \( x', y' \) are scaled on \( d \). The velocity vector \( V' = (u', v', w') \), the temperature \( \theta' \), the time \( t' \), the pressure \( p' \) and the magnetic field vector \( H' = (H'x, H'y, H'z) \) are made dimensionless by scales \( \kappa/d, \beta d, \xi, \rho_o \kappa/d^2 \) and \( \kappa H_o/\eta \). In the Boussinesq approximation the equations that describe the motion of the conducting fluid in an externally impressed oblique magnetic field (in the \( \text{xz} \)-plane) are

\[
\begin{align*}
\nabla V' &= 0, \quad \nabla H' = 0 \\
\frac{1}{\sigma_1} \left[ \frac{\partial}{\partial t'} + (V' \nabla)V' \right] &= -Q \left( 2\sigma_2 \right) (H \cdot \nabla)H' \\
- \nabla \left( P + \frac{Q\sigma_2}{\sigma_1} |H|^2 + Q (H \times \sin \phi + H_z \cos \phi) \right) + Q \left( \cos \phi \frac{\partial}{\partial z} + \sin \phi \frac{\partial}{\partial x} \right) H + R \theta e_z + \nabla^2 V' \quad \ldots \quad (5)
\end{align*}
\]

where

Rayleigh number \( R = \frac{a g \beta d^4}{\nu k} \)

Chandrasekhar number \( Q = \frac{\mu_m H_o^2 d^2}{4\pi \rho_o \nu \kappa} \)

thermal Prandtl number \( \sigma_1 = \frac{\nu}{k} \)

and magnetic Prandtl number \( \sigma_z = \frac{\nu}{\kappa} \)

are dimensionless numbers. We reduce Eqs (5)-(8) to a single equation as follows: The curl of Eq. (6) gives

\[
\begin{align*}
\frac{1}{\sigma_1} \left( \frac{\partial}{\partial t'} - \nabla^2 \right) \omega - Q \left( \cos \phi \frac{\partial}{\partial z} + \sin \phi \frac{\partial}{\partial x} \right) J \\
- (\nabla \theta \times e_z) = \frac{Q \sigma_2}{\sigma_1} [(H \cdot \nabla)J - (J \cdot \nabla)H] \\
- \frac{1}{\sigma_1} \left[ (V' \cdot \nabla) \omega - (\omega \cdot \nabla) V \right] \quad \ldots \quad (9)
\end{align*}
\]

where the vorticity \( \omega = \text{curl } V \), current \( J = \text{curl } H \) and

\[
\begin{align*}
\nabla \times [(V' \cdot \nabla) V] &= (V' \cdot \nabla) \omega - (w \cdot \nabla) V \\
\nabla \times [(H \cdot \nabla) H] &= (H \cdot \nabla) J - (J \cdot \nabla) H
\end{align*}
\]

The curl of Eq. (9) gives, after use of Eq. (5)
\[
\left(1 - \frac{\sigma^2}{\sigma_1^2}\right)\nabla^2 V - Q\left(\cos \phi \frac{\partial}{\partial z} + \sin \phi \frac{\partial}{\partial x}\right)\nabla^2 H + \frac{Q \sigma_2}{\sigma_1} (H \nabla) J - \frac{Q \sigma_2}{\sigma_1} (H \nabla) H \right] \quad \ldots (10)
\]

The \( z \)-components of Eqs (8) and (10) are
\[
\left(\frac{\sigma_2}{\sigma_1} \frac{\partial}{\partial t} - \nabla^2\right) H_z - \frac{\partial w}{\partial z} = \frac{\sigma_2}{\sigma_1} \hat{z} \cdot \left[\nabla \times (V \nabla) - (\omega \nabla)V\right] \quad \ldots (11a)
\]
\[
\left(\frac{1}{\sigma_1} \frac{\partial}{\partial t} - \nabla^2\right) \nabla^2 \omega - Q \left(\cos \phi \frac{\partial}{\partial z} + \sin \phi \frac{\partial}{\partial x}\right)\nabla^2 H_z - \frac{Q \sigma_2}{\sigma_1} \hat{z} \cdot \left[\nabla \times (V \nabla) - (\omega \nabla)V\right]
\]
\[
= \frac{1}{\sigma_1} \left[\nabla \times ((V \nabla) \omega - (\omega \nabla)V]\right] - \frac{Q \sigma_2}{\sigma_1} \hat{z} \cdot \left[(H \nabla) J - (J \nabla) H\right] \quad \ldots (11b)
\]

Eliminating \( \theta \) and \( H_z \) from linear part of Eqs (7), (11a) and (11b), one can get
\[
\begin{align*}
L w &= N \\
L &= \left(\frac{\partial}{\partial t} - \nabla^2\right) \left(\frac{\sigma_2}{\sigma_1} \frac{\partial}{\partial t} - \nabla^2\right) \left(\frac{\sigma_2}{\sigma_1} \frac{\partial}{\partial t} - \nabla^2\right) V^2 \\
- Q \left(\cos \phi \frac{\partial}{\partial z} + \sin \phi \frac{\partial}{\partial x}\right) \frac{\partial}{\partial z} V^2 - R \frac{\sigma_2}{\sigma_1} \frac{\partial}{\partial t} - \nabla^2 \right) \nabla^2 (V \nabla) \theta \right) \\
N &= Q \frac{\sigma_2}{\sigma_1} \left(\frac{\partial}{\partial t} - \nabla^2\right) \nabla^2 (\cos \phi \frac{\partial}{\partial z} + \sin \phi \frac{\partial}{\partial x}) \hat{z} \cdot \left[\nabla \times ((V \nabla) \omega - (\omega \nabla)V]\right] \\
\times [(H \nabla) V - (V \nabla) H]\end{align*} \\
\times \left[\frac{1}{\sigma_1} \nabla \times ((V \nabla) \omega - (\omega \nabla)V]\right] \left[\frac{\sigma_2}{\sigma_1} \frac{\partial}{\partial t} - \nabla^2\right] (V \nabla) \theta \right) \quad \ldots (14)
\]

3 Linear stability analysis

In this section, we study the linear stability analysis of the problem by assuming that axis of the cylindrical rolls is \( x \)-axis, so that \( x \)-dependence disappears from Eqs (5)-(8). We substitute \( w = W(z) e^{\text{i} \omega t} e^{\text{i} q z} \) into linearized version of Eq. (12), viz. \( L w = 0 \) and obtain an equation
\[
\left[(D^2 - q^2 - p) - \frac{p}{\sigma_1}\right] W \times (D^2 - q^2) \left[D^2 - q^2 - \frac{\sigma_2 p}{\sigma_1}\right] W
\]
\[
= -Q \left(D^2 - q^2\right) + Q (D^2 - q^2) \times (D^2 - q^2) - p (\cos \phi D^2) \right] W \quad \ldots (15)
\]

In this paper, we have considered only the idealized stress-free conditions on the surface and vanishing of temperature fluctuations. Thus, \( W = D^2 W = D^4 W = 0 \) at \( z = 0, 1 \). The factor \( W \) and its even derivatives vanish at \( z = 0 \) and \( z = 1 \), which implies that we can assume \( W = \sin \pi z \) as a solution of Eq. (15). Substituting \( W = \sin \pi z \) and \( p = \text{i} \omega \) in Eq. (15), we get
\[
R = \frac{\delta^2}{q^2 (D^4 + \frac{\omega^2 \sigma_2^2}{\sigma_1^2})} \times \left[(D^4 - \frac{\omega^2 \sigma_2^2}{\sigma_1^2}) \omega \sigma_2 \frac{1}{\sigma_1} + Q \pi^2 \cos \phi \right]
\]
\[
+ \omega^2 \sigma_2 \frac{1}{\sigma_1} \left[\sigma_2 \left(D^4 - \frac{\omega^2 \sigma_2^2}{\sigma_1^2}\right) + D^4 \left(1 + \frac{1}{\sigma_1}\right) + Q \pi^2 \cos \phi \right] + i \omega \sigma_2^2 \left(A_2 \omega^2 + A_2^2\right) \quad \ldots (16)
\]

where
\[
\delta^2 = (\pi^2 + q^2), \quad A_1 = \left(\frac{\sigma_2}{\sigma_1}\right)^2 \left(1 + \frac{1}{\sigma_1}\right),
\]
\[
A_2 = \delta^4 \left(1 + \frac{1}{\sigma_1}\right) + Q \pi^2 \cos \phi \left(1 - \frac{\sigma_2}{\sigma_1}\right)
\]

We consider following two cases:

3.1 Stationary convection (\( \omega = 0 \))

Substituting \( \omega = 0 \) in Eq. (16), we get
\[
R_s = \frac{\delta^2 \left(D^4 + Q \pi^2 \cos \phi\right)}{q^2} \quad \ldots (17)
\]

Here, \( R_s \) is the value of \( R \) for the stationary convection. The minimum value of \( R_s \) is obtained for \( q = q_{sc} \), where
2 \left( \frac{q_{sc}}{\pi} \right)^6 + 3 \left( \frac{q_{sc}}{\pi} \right)^4 = 1 + \frac{Q \cos \phi}{\pi^2} \quad \ldots \quad (18)

The wave number is identical to that for the single component fluid, while the threshold for the onset of stationary convection at pitchfork bifurcation is given by Eq. (17) with \( q = q_{sc} \). Thus

\[ R_{sc} = \frac{\delta^2 \left( \frac{q_{sc}^2 + Q \pi^2 \cos \phi}{q_{sc}} \right)}{q_{sc}} \quad \ldots \quad (19) \]

### 3.2 Oscillatory convection \((\omega^2 > 0)\)

For oscillatory convection \( \omega \neq 0 \) and from Eq. (16), \( R \) will be complex. But the physical meaning of \( R \) requires it to be real. The condition that \( R \) is real implies that imaginary part of Eq. (16) is zero, i.e.

\[ A_0 \omega^2 + A_2 = 0 \quad \ldots \quad (20) \]

Value of \( A_2 = 0 \) implies that \( Q = Q_c \) corresponding to Takens-Bogdanov bifurcation point has a \( \omega = 0 \) as a double zero.

Substituting \( W = \sin \pi z \) in Eq. (15), we get a third degree polynomial equation in \( p \) of the form

\[ a_1 p^3 + a_2 p^2 + a_1 p + a_0 = 0 \quad \ldots \quad (21) \]

where

\[ a_1 = \frac{\delta^2 \sigma_2}{\sigma_2^2} \]

\[ a_2 = \delta^4 \left[ \frac{\sigma_2 + \sigma_2 + \sigma_1}{\sigma_1} \right] \]

\[ a_3 = \delta^6 \left( \frac{\sigma_2 + \sigma_1}{\sigma_1} \right) + \frac{\delta^6}{\sigma_1} + \frac{Q \pi^2 \cos \phi - R q^2}{\sigma_2} \frac{\sigma_2}{\sigma_1} \]

\[ a_0 = \delta^8 + \frac{\delta^4 Q \pi^2 \cos \phi - R q^2}{\sigma_2} \delta^2 \quad \ldots \quad (22) \]

Setting \( p = i \omega \) in Eq. (21) and equating its real and imaginary parts to zero, we get

\[ a_1 \omega^2 - a_0 = 0, \quad \ldots \quad (23a) \]

\[ a_2 \omega^2 - a_1 = 0, \quad \ldots \quad (23b) \]

Eliminating \( R \) from Eqs (23a) and (23b), we get

\[ \omega^2 = \frac{\sigma_1}{\sigma_2^2 (1 + \sigma_1)} \left[ Q \pi^2 \cos \phi (\sigma_2 - \sigma_1) - (1 + \sigma_1) \delta^2 \right] \quad \ldots \quad (24) \]

A necessary condition for \( \omega^2 \) to be positive is

\[ \sigma_2 > \sigma_1 \quad \ldots \quad (25) \]

However, it is not sufficient condition and one must have in addition

\[ Q > Q_c = \frac{(1 + \sigma_1) \delta^4}{\pi^2 \cos \phi (\sigma_2 - \sigma_1)} \quad \ldots \quad (26) \]

Substitution for \( \omega^2 \) from Eq. (24) into Eq. (23a), we get

\[ R_0 = \frac{\delta^2 [\delta^4 + Q \pi^2 \cos \phi]}{q^2 \left( \delta^2 \left( \sigma_1 + \sigma_2 + \sigma_1 \right) \sigma_2 \sigma_2^2 (1 + \sigma_1) \right)} \quad \ldots \quad (27) \]

Minimizing \( R_0 \) as a function of \( q \) gives us the critical Rayleigh number, \( R_{sc} \), for oscillatory convection as a function of \( \sigma_1, \sigma_2, Q \) and \( \delta \). Keeping the fluid parameters fixed, if we vary \( Q \), beginning from the region where the convection is oscillatory, we are going to get a critical \( Q = Q_c \), where \( \omega^2 = 0 \) and \( R_0(q) = R_0(q_c) \). \( \partial R_0/\partial q \mid_{q = q_c} \) gives equation in \( q_{sc} \) constant value of \( Q, \sigma_1, \sigma_2, \) viz.

\[ 2 \left( \frac{q_{sc}}{\pi} \right)^6 + 3 \left( \frac{q_{sc}}{\pi} \right)^4 = 1 + \frac{\sigma_2^2 Q \cos \phi}{\pi^2 \left( \sigma_1 + \sigma_2 \right) (1 + \sigma_1)} \quad \ldots \quad (28) \]

From Eq. (21), if \( \omega = 0 \) then \( a_0 = 0 \) and we get stationary convection and \( R \) is determined by putting \( R = R_{sc} \) in \( a_0 = 0 \). Thus, \( \omega = 0 \) and \( a_0 = 0 \) are the conditions for the pitchfork bifurcation corresponding to stationary convection. From Eq. (23), we can have marginal stability if \( \omega^2 = a_1/a_3 (a_1 > 0) \) and

\[ a_2 - a_1 a_3 = 0 \quad \ldots \quad (29) \]

In this case, we get oscillatory convection and \( R_0 \) (the value of \( R \) for the oscillatory convection) is obtained by putting \( R = R_0 \) in the expressions \( a_0, a_1, a_2, a_3 \), of the set of equations [Eq. (22)] into Eq. (28). Takens-Bogdanov bifurcation point is determined by the intersection of the two curves \( a_0 = 0 \) and \( a_1 = 0 \) in \((\sigma_2, R)\)-space. Thus, Takens-Bogdanov bifurcation point corresponds to a double-zero eigenvalue of the linear growth rate. At the co-dimension-two point, we have

\[ R_{sc}(q_{sc}) = R_{sc}(q_{oc}), \quad \text{but} \quad q_{sc} \neq q_{oc} \quad \ldots \quad (29) \]

and at the Takens-Bogdanov bifurcation point, we have

\[ R_0(q_o) = R_0(q_{oc}) = R_0(q_e) \quad \text{and} \quad q_o = q_{oc} = q_e \quad \ldots \quad (30) \]
Fig. 1—Numerically calculated marginal stability curves (—— steady, ----- oscillatory) are plotted in (q, R)-plane for $\sigma_1 = 2$, $\sigma_2 = 6$ and $\phi = 75^\circ$ for values of $Q$ equal to: (a) $10^6$ (b) $10^{12}$ (c) $10^{16}$ (d) $10^{20}$

Fig. 2—Numerically calculated marginal stability curves (—— steady, ----- oscillatory) are plotted in (q, R)-plane for $\sigma_1 = 2$, $\sigma_2 = 10$ and $\phi = 75^\circ$ for values of $Q$ equal to: (a) $10^6$ (b) $10^{12}$ (c) $10^{16}$ (d) $10^{20}$
Eliminating $R$ from $a_0 = a_1 = 0$, we get Takens-Bogdanov bifurcation point at

$$\sigma_{2c} = \frac{\sigma_1}{Q \pi^2 \cos \varphi} (Q \pi^2 \cos \varphi + 2(\pi^2 + q_1^2)^2) \quad \ldots (31)$$

$$Q_c = \frac{(1 + \sigma_1)(\pi^2 + q_1^2)^2}{(\sigma_2 - \sigma_1) \pi^2 \cos \varphi} \quad \ldots (32)$$

Figures 1-3 are plotted in the $(q, R)$-plane. Each solid line and dotted line of Figs 1-3 stands for pitchfork bifurcation and Hopf bifurcation. We have observed the effect of physical parameters like $Q$ and $\sigma_2$ over the onset of both stationary convection and oscillatory convection with the inclination of magnetic field $\phi = 75^\circ$ and $\sigma_1 = 2$. In Fig. 1 when $Q$ increases then the onset of both stationary convection and oscillatory convection increases. This implies magnetic field inhibits the onset of stationary convection and oscillatory convection. In Figs 2 and 3, we showed the effect of other parameter $\sigma_2$. From these figures we can conclude that, when $\sigma_2$ increases then the onset of oscillatory convection decreases. Figure 3(b), shows both primary and secondary bifurcations.

4 Conclusions

In this paper, we have studied magneto-convection due to oblique magnetic field. The onset of stationary convection at the pitchfork bifurcation corresponds to the case of marginal stability. Takens-Bogdanov bifurcation point corresponds to $a_0 = a_1 = 0$. By eliminating the Rayleigh number from $a_0 = a_1 = 0$, we get the value of $Q = Q_c$ or $\sigma_2 = \sigma_{2c}$ given by Eq. (31) or (32). We have also obtained the values of Takens-Bogdanov bifurcation point and co-dimension two-bifurcation points by plotting graphs of neutral curves corresponds to stationary and oscillatory convection for different values of physical parameters relevant to penumbral region of sunspot (Figs 1-3).

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