1. Introduction

Various forms have been suggested for ionospheric conductivity to explain the $S_q$ variation of the terrestrial magnetic field by different workers. In this paper, we have considered the most general form of the conductivity tensor subject to the conditions of the usual coordinate transformations. This has been compared with the tensor conductivity used by earlier authors, and new lines of advancement in the dynamo theory in the ionosphere are indicated.

2. Analysis of Various Formulations

Stewart postulated a large movement of ionized air in the dipole field of the earth generating current system as in the case of a dynamo, at a time when the existence of the ionosphere was not known. Schuster gave a mathematical formulation of the dynamo problem where he used the conductivity as a scalar quantity which is a function of the zenith distance of the sun thereby taking the sun as the source of ionization. He wrote down general relation for integrated conductivity as follows:

$$\rho_1 = (\rho_0) = \sum_{s=0}^{\infty} a_s \cos^s \chi$$

where $\rho$ is a conductivity of the shell

$\chi$ is zenith distance of the sun given by

$$\cos \chi = \sin \delta \cos \theta + \cos \delta \sin \theta \cos (\phi - \phi_0)$$

Later on, attempts were made by Chakrabarty and Pratap wherein the method of solving the dynamo equation was a more exact one based on the orthogonality property of the Legendre harmonics and without imposing any other condition. A new condition was obtained between $a_0$, $a_1$, $a_2$, viz.

$$a_1^2 = \frac{9}{4} a_0 a_2$$

with $\delta$ as the declination of the sun, and $\theta$ the colatitude of the observer. ($\phi - \phi_0$) is the longitude of the observer noted from the noon meridian, $a_s$ are constants to be fitted up with observations. Schuster used the method of successive approximation to solve the dynamo equation by truncating the infinite conductivity series by taking $s = 0$ and $s = 1$. Chapman used the same method but extended the Schuster's analysis by taking series up to third term, $s = 0, 1, 2$ with the condition that the conductivity should be a positive definite function by choosing $a_1 = 3a_0$ and $a_2 = 9/4 a_0$. Chapman wrote the above series as

$$\rho_1 = (\rho_2) = a_0 \left[ 1 + \frac{3}{2} \cos \chi \right]^2$$

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which could make the solution exact.

For the first time Baker and Martyn used a tensor conductivity in the dynamo problem for the ionospheric medium. They used a coordinate system in which the magnetic field formed one of the axes, and wrote the direct, Pederson and Hall conductivities based on the theory of the mean free path developed by Chapman and Cowling. Tensor form of conductivity used by Baker and Martyn can be written in a general coordinate system as:

$$\Sigma = \sigma_1 \delta_{ij} + (\sigma_0 - \sigma_1) B_i B_j + \sigma_2 \epsilon_{ijk} B_k$$

where $\sigma_0$, $\sigma_1$, and $\sigma_2$ are parallel, Pederson and Hall conductivities and $B_i$ are the components of the dipole magnetic field in a Cartesian coordinate system. This conductivity has been generalized to include electric fields as well as the intrinsic position vector $r$. This also gives the components in which the electric and magnetic fields appear together. In any realistic ionosphere system the conductivity should be a function of the electric field besides being an explicit function of $r$. We have given a general form based on the result of a general theorem proved by N. D. Sen Gupta on tensors whose elements are functions of vector fields.
where direct conductivity
\[ \sigma_0 = n_e q_e^2 \left[ \frac{1}{m_e v_e} + \frac{1}{m_i v_i} \right] \]  
(6a)

Pederson conductivity
\[ \sigma_p = n_e q_e^2 \left[ \frac{v_e}{m_e (v_e^2 + \Omega_e^2)} + \frac{v_i}{m_i (v_i^2 + \Omega_i^2)} \right] \]  
(6b)

Hall conductivity
\[ \sigma_h = n_e q_e^2 \left[ \frac{\Omega_{ee}}{m_e (v_e^2 + \Omega_e^2)} - \frac{\Omega_{di}}{m_i (v_i^2 + \Omega_i^2)} \right] \]  
(6c)

with 
\[ \Omega_{ee} = \text{cyclotron frequency of electron} \]
\[ \Omega_{ii} = \text{cyclotron frequency of ion} \]
\[ v_i = \text{collision frequency of ion with neutral} \]
\[ v_e = \text{collision frequency of electron with neutral} \]
\[ n_e = \text{density of electron} \]
\[ q_e = \text{charge of electron in e.m.u.} \]

These forms have been used very extensively in recent years in constructing models of electrojet\(^{18}\) as well as \(S_e\) current system\(^{14}\). This is discussed in detail by Chapman\(^{16}\).

The above discussion brings out the following features:

(i) The scalar conductivity taken as a series [Eq. (1)] explains the general pattern by a dynamo theory based on a \(Y_k\) wind system.\(^7\) The fact that a semi-diurnal wind system generates a diurnal magnetic field variation of the form \(Y_k\) shows the importance of the space dependence of the conductivity function, since it is this that converts the second harmonic to first harmonic by the method of beats. This conductivity, however, is independent of the magnetic or electric field strengths.

(ii) Tensor conductivity given by Eq. (4), however, depends on the magnetic field. The number density as well as collision frequency appear through the functions \(\alpha_0, \alpha_1\) and \(\alpha_2\) as has been obtained by Baker and Martyn\(^{18}\) or by Chapman and Cowling.\(^11\) While this explains the enhancement in the equatorial region, as observed in the electrojet phenomena, it is indeed a poor model in explaining the global feature of \(S_e\) field.

3. Generalized Conductivity Tensor

We, therefore, come to the conclusion that (i) the conductivity must be tensorial in form, and (ii) its elements must be space-time dependent as well as dependent on the field quantities. One, therefore, is faced with the problem: what is the general form of a conductivity whose elements are functions of space-time and fields, and secondly, can we derive this tensor consistently from a plasma theory?

Sen Gupta\(^{18}\) answered the first part of the question, viz. that the most general form of a tensor whose elements are functions of a vector field \(A\), and which obeys the usual coordinate transformation laws, is given by

\[ T_{ij} = \phi_0 \delta_{ij} + \phi_1 A_i A_j + \phi_2 A_{ij} A_k \]  
(7)

The above tensor consists of three parts, the first being a diagonal tensor, the second, the symmetric part and the third, the antisymmetric one. The functions \(\phi_0, \phi_1\) and \(\phi_2\) are arbitrary functions constructed out of the scalar formed from the vector field \(A\). If there are two vector fields \(A\) and \(B\), then the tensor \(\tilde{T}\) is given by

\[ T_{ij} = \phi_0 \delta_{ij} + \phi_1 A_i B_j + \phi_2 A_{ij} B_k \]  
(8)

One can easily see that Eq. (5) is a special case of Eq. (8) when the vector field is given by the magnetic field \(B\) of Eq. (8). But, in general, in addition to this, dynamo electric field \(E\) as well as the velocity field \(V\) besides the intrinsic field \(r\), exist. Thus, the general conductivity tensor should be of the form

\[ T_{ij} = \phi_0 \delta_{ij} + \phi_1 (r_i B_j + B_{ij} B_k) + \phi_2 (r_i B_j + B_{ij} B_k) \]  
(9)

where as has already been mentioned, \(\phi_0, \phi_1, \phi_2\) a_i, b_i and c_i are all arbitrary scalar functions of the scalars constructed out of different vector fields.

4. Discussion and Conclusion

One may observe that if we take Eq.(9) as the conductivity tensor, then the tensor is an explicit function of the vector fields as they appear in the various quantities and they are also functions of these fields through the scalar functions. These functions could also be functions of the other parameters in the problem such as collision frequency, temperature, plasma frequency, cyclotron frequency, etc. One can
To identify the rest of the quantities, one will have to obtain a "generalized Ohm's law" in a plasma wherein all the non-linear processes are considered, which is evident from the fact that, then and only then, we can obtain the field dependent conductivity and resistivity tensors. This theory, however, is nonexistent today. We can nevertheless use this above conductivity in developing a three-dimensional dynamo theory in the ionospheric region and include the dynamics of vertical velocity fields in the near future. On the other hand, in developing a generalized Ohm's law taking into account the non-linear momentum transfer, one could look for such functional form for the conductivity tensor as given in Eq. (9).

References