A Non-linear Theory of Gravitation-induced Rayleigh-Taylor Instability Mechanism & the Equatorial Spread-F

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The non-linear evolution of the collisional, gravitation-induced Rayleigh-Taylor (R-T) instability in the equatorial F-region has been studied, taking into account the finite Larmor radius effects and the complete ion inertial term in ion equation of motion. A special class of coherent weakly non-linear modes as solutions to the wave equation describing R-T instability driven modes, is obtained. It is shown that the R-T modes in the equatorial F-layer can evolve into coherent, non-linear, almost sinusoidal, stationary wave structures. These structures are found to travel with a constant phase velocity and to have distorted sinusoidal shapes. The present results show a reasonably good agreement with many of the recent rocket and satellite observations of the equatorial spread-F irregularities.

1. Introduction

The well known phenomenon of equatorial spread-F has, widely, been attributed to the field-aligned electron density irregularities occurring in the equatorial F-region. The spread-F has been the subject of both theoretical and experimental investigations by many workers in the recent past. Yet the processes responsible for producing these irregularities are not fully understood. Therefore, an attempt has now been made to investigate a non-linear collisional gravitation-induced Rayleigh-Taylor (R-T) instability as the possible mechanism for their generation.

The equatorial spread-F is essentially a nighttime phenomenon associated with a post sunset rise of the F-layer. The most important and obvious source of free energy for deriving an instability on the bottomside of the nighttime F-layer is the sharp vertical plasma density gradient. Therefore, the F-region plasma after sunset resembles a slab of inhomogeneous plasma, and such a plasma is known to be driven to become unstable due to R-T instability. Dungey was, in fact, the first to suggest that the source of equatorial F-region irregularities could be the gravitational instability of the underside of the F-layer. Many of the observed properties of long wavelength irregularities \( \lambda_s > \rho_i \) (ion Larmor radius) below the F-region maximum can actually be explained in terms of this gravitation-induced R-T instability. But there are still many features of spread-F irregularities observed by satellites and rockets which need to be explained by a satisfactory non-linear theory.

In all the earlier linear and non-linear theories of spread-F irregularities, the linear convection part of ion inertial term has been neglected. However, the convectional motion of plasma in the equatorial F-region appears to be quite important, which can give a significant contribution to the irregularity formation as suggested by Dungey as well as to the stabilization of the long wavelength R-T instability driven modes in the F-region. Furthermore, the phase velocity \( u \) of the waves may be comparable to or even smaller than ion diamagnetic drift \( V_L \). For such a situation, it, therefore, becomes necessary to include the linear convective part \( (V_0 \cdot \nabla)V \) in ion equation of motion where \( V_0 = V_g + V_L \) (\( V_g \) is the gravitational drift and \( V_L < V_g \)), whereas earlier workers have overlooked this term in spite of assuming \( u < V_L \) and this may make their conclusions rather inconsistent. This paper deals with a non-linear theory of the collisional R-T instability accounting for FLR and ion viscous effects, which differs from the earlier theories in the sense that it includes in the ion equation of motion the convective part also.

2. Basic Equations

The F-region plasma can, adequately, be described by the macroscopic fluid equations for the perturbation wavelengths of our interest which are assumed to be greater than either ion gyro-radius or ion mean free path. We choose the following rectangular coordinate system appropriate to the equatorial F-region. Z-axis points along the direction of earth’s magnetic field, Y-axis is taken along the west-east direction and X-axis points vertically upwards. The background electron density gradient is thus along the X-axis and anti-parallel to the acceleration due to gravity \( g \). In the frame of two-fluid theory, the equatorial F-region plasma is, therefore, governed by the continuity and momentum transfer...
equations for electrons and ions which are given by:

\[ \frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 \quad \ldots (1) \]

\[ - \frac{e}{m} \left( \mathbf{E} - \frac{\mathbf{v} \times \mathbf{B}}{c} \right) - \frac{KT_e}{m} \nabla n = 0 \quad \ldots (2) \]

\[ \frac{\partial N}{\partial t} + \nabla \cdot (N\mathbf{v}) = 0 \quad \ldots (3) \]

\[ \frac{\partial \mathbf{V}}{\partial t} + \left( \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{e}{M} \left( \mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} \right) - \nabla \mathbf{V} + \mathbf{g} \]

\[ - \frac{KT_i}{M} \nabla N + \mu \nabla^2 N \quad \ldots (4) \]

where \( n \) and \( N \) are the number densities of electrons and ions, respectively, \( \mathbf{E} \) is the electric field, \( \mathbf{v} \) and \( \mathbf{V} \) represent the electron and ion velocities, \( m \) and \( M \) refer to the electron and ion masses, respectively, \( \mu \) is the coefficient of ion viscosity and \( \mathbf{v} \) is the ion-neutral collision frequency. All other symbols have their conventional meanings.

2.1 Basic Assumptions

In writing down the above equations and in subsequent analysis, the following assumptions and approximation are made. (i) The electron collision frequency \( \nu_e \) is neglected because \( \nu_e < \Omega_e \), the electron gyrofrequency. (ii) The electron inertia has been ignored since \( \Omega_e \) is much larger than the Doppler-shifted wave frequency. (iii) The wave electric field is essentially assumed to be electrostatic, i.e. \( \mathbf{E} = - \nabla \phi \) where \( \phi \) is the electrostatic potential. (iv) The electrons and ions are assumed to be isothermal. (v) The ion drag, which is unimportant at night,22 is neglected. (vi) The recombination is assumed to be negligible, which is valid for the processes that occur faster than the several hour time scale of the nighttime F-region. (vii) The waves are assumed to be propagated exactly perpendicular to the magnetic field, since these waves experience least damping due to diffusion. (viii) The neutrals are considered completely at rest.

Since we are interested in long wavelength R-T modes, we invoke quasineutrality condition, namely,

\[ n \approx N \quad \ldots (5) \]

2.2 Equilibrium Configuration

We first assume here that the frame of reference considered above is moving with \( \mathbf{E}_0 \times \mathbf{B} \) drift. This assumption of transformation to a \( \mathbf{E}_0 \times \mathbf{B} \) drifting frame of reference enables us to neglect in the analysis the effects of the equilibrium dc electric field \( \mathbf{E}_0 \). In zero-order stationary equilibrium state, Eqs. (1)-(4) yield the electron and ion zero-order equilibrium drift velocities in the horizontal and vertical directions as follows.

\[ v_{\alpha x} \approx 0 ; \quad V_{\alpha x} = - (n/\Omega_e) V_L \quad \ldots (6) \]

\[ v_{\alpha y} \approx V_L ; \quad V_{\alpha y} = g/\Omega_i + V_L \quad \ldots (7) \]

where \( v_{\alpha x} = v_{th, \alpha}/L \) and \( V_{\alpha x} = V_{th, \alpha}/L \) are the electron and ion diamagnetic drifts, respectively, \( v_{th} = v_{th}/\Omega_e \) and \( V_{th} = V_{th}/\Omega_i \) are the electron and ion Larmor radii, respectively. \( v_{th} = (KT_e/m)^{1/2} \) and \( V_{th} = (KT_i/M)^{1/2} \) are, respectively, the electron and ion thermal velocities and \( L \) is the background density scale length defined by \( L^{-1} = (1/n_0) \, d n_0/d x \).

Thus, the FLR effects are introduced through \( V_L \) and \( v_{\alpha x} \). Both the ion collision frequency \( \nu_i \) and the ion diamagnetic drift \( V_L \) decrease with increasing altitude. Therefore, the equilibrium ion vertical drift velocity becomes vanishingly small, resulting into ambipolar diffusion of the particles at great heights. The ion viscous diffusion flux along the density gradient, which is antiparallel to \( (\nabla N/O_\nu) \), is also negligible and has not been considered \( (\nabla N/O_\nu < 10^{-8}) \), and \( V_L \) is typically \( \gg 1 \) m/sec above 250 km.

3. Non-linear Wave Equation for R-T Instability Driven Modes

Let us now define each variable \( q \) consisting of two parts:

\[ q = q_0 + \tilde{q} \quad \ldots (8) \]

where \( q_0 \) denotes the equilibrium value and \( \tilde{q} \) is the horizontally propagating low frequency perturbation of the form \( \exp \{i (k y - \omega t) \} \), where \( \omega \) is the wave frequency and \( k \) the wave number. Before deriving the non-linear wave equation for R-T modes, in the following we will first point out briefly the importance of the conventional part of ion inertial term.

Having neglected the FLR corrections which only lead to the stabilization effects, Jain19 and Jain and Das30 have obtained the linear dispersion relations both for the collisionless \( (\nu_i = 0) \) and collisional \( (\nu_i \neq 0) \) cases. They have shown that the R-T modes are not purely growing ones because there exists a real positive constant phase velocity rather than a zero phase velocity as obtained by earlier authors. The magnitude of the phase velocity is found to be equal to half the ion gravitational drift velocity \( V_g = g/\Omega_i \) in the collisionless limit and equal to \( g V_L/2 L \) in the collisional limit. It was shown by these authors that the growth rate of collisional R-T modes is in agreement with that obtained by Balsley et al.7 and Hudson and Kennel53,16 while in the collisionless case there is reduction in the growth rate obtained by all earlier workers, by a factor equal to \( k L^2 (g/L)^{1/2}/8 \Omega_i \) rather than a purely growing term. These workers emphasized that these differences arise due to the inclusion of the conventional
term (\(V_0 \cdot \nabla\)) \(\tilde{V}\) which may thus contribute to the stabilization of R-T modes for a certain range of wave numbers even in the absence of FLR corrections and neglecting \(w_1\).

It can here be noted, however, that the phase velocity in collisional case is reduced down by a factor of \(2g^2/(L \omega)\) from that obtained in the collisionless limit and approaches its collisionless limit (as shown by these workers) at higher altitudes where \(v_i\) is negligibly small. Nevertheless, the phase velocity of R-T modes remains smaller than the ion gravitational drift \(V_0\) (\(\approx 10\) m/sec) in the regions of interest. Chaturvedi and Kaw17 and all other earlier workers have neglected the term (\(V_0 \cdot \nabla\)) \(\tilde{V}\) in ion equation of motion inspite of the fact that \(u\) is smaller than \(V_0\). This seems to be rather inconsistent with their results which are essentially obtained on the basis of the assumption that \(u < V_L\) or \(V_0\). It has, therefore, become quite obvious now that one should include the convective part in ion inertia in a satisfactory theory for spread-F irregularities.

Substituting Eq. (8) into Eqs.(1)-(4) and using the zero-order equilibrium state \(q_0\) and the condition in Eq. (5), we get the equations of continuity and motions for both the electron- and ion-perturbed quantities, \(\tilde{q}\)'s. Now, combining equations of continuity and motion for electrons, we get the equation for electron density perturbation as

\[
\frac{\partial n}{\partial t} - \frac{n_0}{\omega_L} \left( \frac{c}{B} \cdot \frac{\partial \tilde{\phi}}{\partial y} \right) - \frac{n_0}{\omega_L} \left( \frac{c}{B} \tilde{\rho} + \frac{g}{\omega_L} \tilde{h} \right) = 0 \quad \ldots (9)
\]

which shows that the density changes locally because of \(E \times B\) motion of electrons along the direction of density gradient (\(\nabla n = -\omega n_0/L\)). The solution of the ion equation of motion for the components of ion velocity \(\tilde{V}_x\) and \(\tilde{V}_y\), and substitution of these two values into the ion equation for continuity, would yield the equation for ion density change due to motions of ions as given below:

\[
\frac{\partial h}{\partial t} + \frac{n_0 V_L}{\omega_L} \left( \frac{\partial}{\partial t} \log \left( \frac{n}{n_0} \right) - \frac{n_0}{\omega_L} \frac{c}{B} \frac{\partial \tilde{\phi}}{\partial y} + \frac{g}{\omega_L} \frac{\partial h}{\partial y} \right)
- \frac{1}{\omega_L} \left[ \left( \frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial y} - \mu \frac{\partial^2}{\partial y^2} \right) \frac{\partial}{\partial y} \left( V_L \frac{\partial h}{\partial y} \right) + \frac{c}{B} \frac{n}{n_0} \tilde{\rho} + V_0 \frac{\partial}{\partial y} \left( V_L \frac{\partial h}{\partial y} \right) + \frac{c}{B} \frac{n}{n_0} \frac{\partial \tilde{h}}{\partial y} \right]
- \frac{\partial}{\partial y} \left( \left( \frac{c}{B} \frac{\partial h}{\partial y} + \frac{\partial}{\partial y} \left( V_L \frac{\partial h}{\partial y} \right) \right) \frac{\partial}{\partial t} V_0 + \frac{\partial}{\partial y} \right)
- p^2 \frac{\partial^2}{\partial y^2} \tilde{h} = 0 \quad \ldots (10)
\]

where the terms which contain the square and product of \(\omega/\Omega_i, k/\Omega_i\) and \(w_i/\Omega_i\) and other smaller terms have been neglected in comparison with other leading terms. In Eq. (10), the third term describes the generation of density fluctuation because of \(E \times B\) effect in the direction of perpendicular density gradient similar to the case of electrons. The fourth term represents the density fluctuation arising because of the gravitational drift of ions, and the last term gives the contribution due to the effects of finite inertia, ion collisions and ion viscosity.

Combining Eq. (9) for electrons and Eq. (10) for ions, we obtain the non-linear partial differential equation in \(y\) and \(t\) for the R-T modes. The solution of this equation will describe the non-linear evolution of R-T modes. However, it is quite difficult to solve this non-linear equation as such. A more simpler equation can be derived by making a suitable transformation such that all the oscillating quantities are functions of only \(s - ut\), where \(s\) is the coordinate along the direction of wave propagation and \(u\) is the phase velocity \(u = \omega/k\). In the case of periodic modes it is convenient to introduce the phase of oscillation as

\[
\xi = ky - \omega t \quad \ldots (11)
\]

We assume that \(u < V_0\) consistent with earlier discussions and that \(n = n_0 < 1\) in agreement with the observations.14 The use of the transformation Eq. (11) reduces the partial differential equation in \(y\) and \(t\) into an ordinary differential equation in \(\xi\), which after one integration and some algebra yields the following non-linear wave equation for R-T modes:

\[
\frac{d^2 \eta}{d\xi^2} + \frac{V_L}{u - V_L} \left( \frac{1}{1 + m \eta} - \frac{1}{(1 + m \eta) (1 + \eta)} \right) \left( \frac{d\eta}{d\xi} \right)^2
- \frac{\eta k(V_0 - u)}{u - V_L} \frac{d\eta}{d\xi} + \frac{p^2}{1 - m \eta} \frac{\eta}{1 + \eta} \frac{d\eta}{d\xi} + \frac{1}{1 + m \eta} \frac{\eta}{(1 + \eta) (1 + \eta)} \frac{d\eta}{d\xi} = 0 \quad \ldots (12)
\]

where

\[
m_0 = u/(u - V_L) \quad m = uV_L/2gL \quad p^2 = g/[k^2 L (V_0 - u) (u - V_L)] \quad \ldots (13)
\]

where we have, for simplicity, ignored \(\mu\) which is also found to be much smaller in magnitude compared to other terms below the equatorial F-region maximum.

Eq. (12) is the exact non-linear wave equation describing the low-frequency and long-wavelength R-T modes generated on the underside of the F-region...
maximum of the equatorial ionosphere. One can now readily see that the periodic solutions of this wave equation are possible only when the fourth term is a positive quantity, i.e. $u - V_L > 0$ with $u < V_0$. The equation includes the contribution due to FLR corrections as well as that due to the linear convolutional part $(V_0 - V) \sim V$ appearing in ion motion.

4. Non-linear Solution of the Wave Equation

For weakly non-linear modes, $\gamma \ll 1$ and the wave equation [Eq. (12)] for weakly non-linear R-T modes can, therefore, be written as

$$\frac{d^2 \eta}{d \xi^2} + \frac{V_L}{u - V_L} \left(1 + m_0 \gamma \right) \left(1 + \gamma \right) \left(\frac{d \eta}{d \xi}\right)^2 + \frac{\nu_1}{k (V_0 - u)} \frac{d \eta}{d \xi} + \frac{p^2}{1 + m_0 \gamma} \left(1 + \frac{1}{1 + m_0 \gamma} \right)^2 \eta^2 = 0 \quad \ldots (14)$$

In the weakly non-linear regime, the amplitude and the phase of the wave vary slowly with $\xi$. We use the method of averaging of Bogoliubov and Mitropolsky to solve Eq. (14). Let the solution be

$$\eta = A \cos \left( p_0^2 + \phi \right) \ldots (15)$$

where $A$ and $\Phi$ are slowly varying functions of $\xi$. Following Bogoliubov and Mitropolsky, we obtain the dependence of $A$ and $\Phi$ on $\xi$ as follows.

$$\frac{d A}{d \xi} = - \frac{\nu_1 A}{2 k (V_0 - u)} \left(1 + m_0 A^2/2\right) \quad \ldots (16)$$

$$\frac{d \Phi}{d \xi} = - \frac{1}{2} \frac{A p}{1 + m_0 A^2/2} \left(1 - \frac{3}{8} m_0 \left(1 - \frac{u V_L}{\epsilon_m g L}\right) A \right) \quad \ldots (17)$$

In the non-linear stationary regime, the amplitude gets saturated and does not vary with $\xi$, i.e. the saturated amplitude is given by

$$\frac{d A}{d \xi} = - \frac{\nu_1 A}{2 k (V_0 - u)} \left(1 + \frac{u}{u - V_L} A^2 \right)^{-1} \times \left(1 + \frac{u}{u - V_L} A^2 \right)^{-1} = 0 \quad \ldots (18)$$

The saturated amplitude $A_0$ can be obtained by neglecting the terms of order higher than $A^2$ in the expansion of $\left(1 + \frac{u}{u - V_L} A^2 \right)^{-1}$ and is given by

$$1 - \frac{u_0}{u_0 - V_L} \cdot \frac{A_0^2}{4} = 0, \quad \text{i.e.} \quad u_0 = V_L \left(1 + A_0^2/4\right) \quad \ldots (19)$$

where $u_0$ is the value of $u$ at saturation amplitude $A_0$. With this stationary amplitude, Eq. (14) is satisfied by the oscillatory solution of the type

$$\eta = A_0 \cos \left(k \gamma - \omega t + \phi \right) \quad \ldots (20)$$

if $p^2 = 1$, i.e. $(V_0 - u) (u - V_L) = g/\gamma L \quad \ldots (21)$

Eq. (17) determines how the phase of the waves changes slowly with $\xi$ in non-linear state. When the amplitude of the waves becomes stationary, the change in phase can be obtained from the following simplified equation

$$\frac{d \Phi}{d \xi} = \frac{p}{2} \left[1 - \frac{3}{8} \cdot \frac{u}{u - V_L} \left(1 - \frac{u (u - V_L)}{6 g L}\right) A_0 \right.$$

$$- \left. \frac{u}{u - V_L} \cdot \frac{A_0^2}{4}\right] A_0 \quad \ldots (22)$$

where $\Phi_0$ corresponds to the value of $\Phi$ in non-linear regime when the amplitude attains a constant value $A_0$. This corresponds to a situation in which coherent, small amplitude, stationary, almost sinusoidal travelling oscillations satisfying both Eqs (19) and (21) exist in the system which supports R-T modes. Eq. (19) shows that the phase velocity of the waves depends upon the non-linear amplitude $A_0$ and also becomes greater than $V_L$ by a factor of $V_L A_0^2/4$ as the amplitude of the modes increases as a result of linear growth. This implies that the R-T modes will become stationary only when $u - V_L \approx V_L A_0^2/4$, i.e. $u - V_L > 0$, which is also the condition for periodicity.

Thus we find that the FLR corrections contribute to the real frequency of the wave and reduce the growth rate [as can be seen from the dispersion relation in Eq. (21)], making the modes more stable. We also find that for the regions of our interest, the leading non-linear term in Eq. (14) is one which is proportional to $(d^2 \eta / d \xi^2)^2$ and depends upon $V_L$. This clearly shows that the FLR corrections seem to be responsible for producing the important non-linearities which, in turn, lead to the stabilization of R-T modes through the non-linear interactions of the waves.\(^{19,20}\)

The mechanism of the stabilization of growing R-T modes may thus be understood physically in the following manner. The fundamental unstable long wavelength R-T modes are first generated and then grow in amplitude. When the amplitude becomes sufficiently large, the higher harmonics are produced which can get coupled with the fundamental through non-linear interactions. The FLR effects reduce the growth rate and the non-linear interactions contribute to the phase velocity. When the phase velocity becomes greater than $V_L$ in the non-linear regime, the non-linear interactions lead to the dissipation of energy associated with the fundamental to its higher harmonics which have shorter wavelengths. As a result the large wavelength waves would become stable because there is a cut-off in growth for higher harmonics below a certain wavelength due to FLR effects.
5. Discussion and Conclusions

We have shown that for the situation \( V_L < u < V_A \)
which is valid in the regions of interest, the coherent,
almost sinusoidal, travelling stationary non-linear
solutions of the equations describing R-T instability
driven modes are found to exist in the equatorial
F-region. These modes have a constant phase velo-
city [given by Eq. (19)] arising due to non-linear
interactions (appearing through FLR effects) with
the higher harmonics generated by the fundamental
unstable R-T modes.

It appears quite certain now that the Rayleigh-
Taylor instability must be the main source of the
irregularities of large scale sizes (> \( r_{li} \)) in the equa-
 torial spread-F region. The recent satellite observa-
tions\(^{16-18} \) do indeed provide the experimental evi-
dence for such non-linear travelling wavelike struc-
tures of equatorial spread-F irregularities over a
large range of scale sizes (\( < 70 \text{ m} \) to 20 km). Dyson
et al.\(^{11} \) observed that the wave form of the irregu-
larities is composed of the fundamental and harmo-
nics. The scale sizes of the fundamental is around
7.5 km which seems to agree well with the typical
wavelength of an R-T mode strongly excited in the
equatorial F-region. Presumably, this is the scale
size at which the energy dissipation from the funda-
mental to higher harmonics occurs due to non-linear
interactions between them, and thereby, giving
energy to smaller irregularities with scale sizes down
70 m or even less.

It is obvious from Eq. (22), however, that if the
non-linear stationary amplitude is sufficiently large,
itis destroys the phase-coherence which the growing
modes display when the amplitude is small, i.e. the
phase of the waves changes with \( \xi \). This, in turn,
distorts the model structure (sinusoidal) as energy
is interchanged between fundamental and harmonics,
indicating that the modes evolve finally into distor-
ted sinusoidal shapes rather than pure sinusoidal
forms. The experimental evidence in favour of this
result seems to arise from recent rocket observa-
tions.\(^{12-18} \) These workers reported that the equatorial
spread-F irregularities following the onset, develop-
ed into coherent, steepened wavelike structures in
a time period of \( \sim 30 \text{ min} \), in good agreement with
our results. The typical growth time of \( \sim 10^3 \text{ sec} \)
for the R-T modes estimated in the F-region, also
seems to agree fairly well with the observed growth
time of the irregularities.

In conclusion, the fact that such non-linear
structures are observed only in the equatorial
spread-F region is of considerable significance, since
it ties up well with the views that the R-T instability
discussed here is the main source of the irregulari-
ties and that our results appear to show a good
agreement with some of the recent rocket and
satellite observations of spread-F irregularities.

References