Propagation of whistler waves and their instability in the magnetosphere

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The dispersion equation for whistler mode waves in the anisotropic plasma in the presence of parallel electrostatic field is derived. Using the realistic models of magnetospheric plasma and field parameters, the refractive index surfaces for whistler mode waves have been constructed. The presence of parallel electric field and anisotropy deform the refractive index surfaces and produce defocussing of whistler waves. The presence of parallel electric field induces an instability in the whistler mode waves which are found to depend on the magnitude and direction of the electrostatic field. It is concluded that the analysis of whistler wave records received on the ground is capable of accounting for the role of plasma anisotropy and the presence of parallel electric fields.

1 Introduction
The magnetosphere extends many earth radii above the ionosphere. It is the region in space where the earth's magnetic field has dominant control over the dynamics of charged particles. The presence of these charged particles in turn determines the characteristics of the different electromagnetic and electrostatic mode waves which can propagate in the magnetosphere. Various ground-based and in situ satellite measurements have made important contributions in the understanding of the characteristics of plasma waves (EIC, LH, ULF, ELF, VLF, LF and HF) and in determining the characteristics of the plasma throughout the magnetosphere. The propagation of very low frequency electromagnetic waves in the whistler mode through the magnetosphere is well known. The propagation and instabilities of whistler mode waves in the hot magnetospheric plasma has often been studied by many researchers by ray tracings, numerical methods and analytically with different geophysical conditions. The instabilities are used to explain various geophysical phenomena in the magnetospheric plasma such as particle precipitations, formation of auroras, natural and artificial VLF emissions. The rocket and satellite observations of charged particles fluxes and their angular distributions have undoubtedly proved the existence of electrostatic field parallel to the geomagnetic field in the ionosphere and magnetosphere. The measured electrostatic field parallel to the geomagnetic field lines is reported, at times, of the order of tens mV/m. The re-investigation of wave propagation and wave-particle interactions in the presence of parallel electrostatic field has been carried out by many workers. The charged particle density inhomogeneities are also well known source of guidance of whistlers.

It this paper we have carried out the detailed theoretical study of refractive index surfaces along the geomagnetic field lines in the presence of plasma anisotropy and parallel electric field. It is well known that whistlers spend maximum time in the equatorial region of the magnetosphere. The wave-particle interactions in this region play very important role as compared to the ionosphere. Therefore, in this study we have studied the resulting growth/decay of whistlers by accounting for the role of temperature anisotropy of the electron distribution functions and weak parallel electrostatic field in the equatorial region. The shape and the dependence of refractive index surfaces and growth/decay of whistlers on various plasma and field parameters have been studied. The results have been compared with the earlier results of Misra et al., Das and Singh, and Singh and Singh. These features of refractive index surfaces and growth/decay rate of whistlers could play an important role in the suppression or enhancement of whistler's intensity in the certain frequency ranges as depicted, at times, in the whistler sonograms.

2 Dispersion equation
The dispersion equation for whistler mode waves propagating through the magnetoplasma with and without electrostatic field parallel to geomagnetic field lines has been derived by many investigators. Recently, Singh and Singh have der-
derived the dispersion equation of whistler mode waves in the presence of parallel electric field for a double Maxwellian velocity distribution function consisting of thermal and energetic electrons using the well known quasi-longitudinal approximation. Following Singh and Singh\(^3\), in this paper, we have derived the dispersion equation for anisotropic velocity distribution functions in the presence of parallel electrostatic field. This parallel electrostatic field is assumed to be sufficiently weak such that drift velocity of the charged particles is much smaller than the phase velocity of whistler mode waves. Making use of quasi-longitudinal approximation as detailed by Singh and Singh\(^4\), the simplified dispersion equation for whistler mode waves is written as

\[
n^2 = \frac{c^2 k^2}{\omega^2} = 1 + \frac{\delta^2}{\Omega} \left(\frac{1}{s + jz} + \frac{k^2 V_{T_i}^2}{2 \omega_i^2 (s + jz)^2}\right)
\]

where

\[
\delta = \frac{\omega_p}{\omega_i}, \quad \Omega = \frac{\omega}{\omega_i}, \quad z = \frac{\nu}{\omega_i}, \quad k_i = k + j K_i, \quad K_i = \frac{e E_0}{K T_i}
\]

\[s = (\cos \theta - \Omega), \quad A_r = \left(\frac{V_{T_i}}{V_{T_i}} - 1\right)
\]

and \(V_{T_i}\) and \(V_{T_i}\) are respectively the parallel and perpendicular components of thermal speed of the particle, \(A_r\) is the anisotropy factor, \(\omega\) the angular frequency of the whistler mode waves, \(\omega_i\) the angular plasma frequency, \(\omega_i\) the angular gyrofrequency, \(\nu\) the velocity independent collision frequency, \(\theta\) the angle between wave vector \(k\) and the static electric field \(E_0\) (\(E_0\) is parallel to the magnetic field) and \(K\) the Boltzmann constant. The refractive index is complex, in general, and is written as \(n = \mu + j \chi\), where \(\mu\) and \(\chi\) respectively refer to real and imaginary parts. Rationalizing Eq. (1) and separating real and imaginary parts of the complex permittivity, we obtain

\[
(\mu + j \chi)^2 = 1 + \frac{\delta^2 s}{\Omega (s^2 + z^2)} \times \left[1 + \frac{V_{T_i}^2 (k^2 - K_i^2) s^2 + 6 k K_i s z V_{T_i}^2}{2 \omega_i^2 (s^2 + z^2)^2}\right]
\]

\[
- \frac{A_r V_{T_i}^2}{2 \omega_i^2 \Omega} \left(\frac{(k^2 - K_i^2) s + 4 k K_i z}{(s^2 + z^2)}\right)
\]

\[
-j \left[\frac{\delta^2 z}{(s^2 + z^2)} \left(1 + \frac{3 V_{T_i}^2 (k^2 - K_i^2) s^2}{2 \omega_i^2 (s^2 + z^2)^2}\right) + \frac{A_r V_{T_i}^2}{2 \omega_i^2 \Omega} \left(\frac{s (k^2 - K_i^2)}{s^2 + z^2}\right)\right]
\]

\[
+ \frac{A_r V_{T_i}^2}{2 \omega_i^2 \Omega} \left(\frac{s (k^2 - K_i^2)}{s^2 + z^2}\right)
\]

\[
\Omega \omega_i^2 (s^2 + z^2)^2
\]

\[
- \frac{A_r V_{T_i}^2}{2 \omega_i^2 \Omega} \left(\frac{s (k^2 - K_i^2)}{s^2 + z^2}\right)
\]

\[
= \epsilon_1 - j \epsilon_2 \quad \ldots (2)
\]

where \(\epsilon_1\) and \(\epsilon_2\) are the real and imaginary parts of the complex permittivity. The real and imaginary parts of refractive index are thus separated and written as

\[
\mu = \frac{1}{\sqrt{2}} \left[\epsilon_1 + (\epsilon_1^2 + \epsilon_2^2)^{1/2}\right]^{1/2}
\]

\[
\chi = \frac{1}{\sqrt{2}} \left[- \epsilon_1 + (\epsilon_1^2 + \epsilon_2^2)^{1/2}\right]^{1/2} \quad \ldots (3)
\]

where

\[
\epsilon_1 = 1 + \frac{\delta^2 s}{\Omega (s^2 + z^2)} \times \left[1 + \frac{V_{T_i}^2 (k^2 - K_i^2) s^2 + 6 k K_i s z V_{T_i}^2}{2 \omega_i^2 (s^2 + z^2)^2}\right]
\]

\[
- \frac{A_r V_{T_i}^2}{2 \omega_i^2 \Omega} \left(\frac{(k^2 - K_i^2) s + 4 k K_i z}{(s^2 + z^2)}\right)
\]

\[
+ \frac{A_r V_{T_i}^2}{2 \omega_i^2 \Omega} \left(\frac{s (k^2 - K_i^2)}{s^2 + z^2}\right)
\]

\[
\frac{\delta^2 z}{(s^2 + z^2)} \left(1 + \frac{3 V_{T_i}^2 (k^2 - K_i^2) s^2}{2 \omega_i^2 (s^2 + z^2)^2}\right)
\]

\[
- \frac{A_r V_{T_i}^2}{2 \omega_i^2 \Omega} \left(\frac{s (k^2 - K_i^2)}{s^2 + z^2}\right)
\]

\[
\Omega \omega_i^2 (s^2 + z^2)^2
\]

\[
- \frac{A_r V_{T_i}^2}{2 \omega_i^2 \Omega} \left(\frac{s (k^2 - K_i^2)}{s^2 + z^2}\right)
\]

\[
= \epsilon_1 - j \epsilon_2 \quad \ldots (4)
\]

\[
\epsilon_2 = \frac{\delta^2 z}{\Omega (s^2 + z^2)} \left[1 + \frac{3 V_{T_i}^2 (k^2 - K_i^2) s^2}{2 \omega_i^2 (s^2 + z^2)^2}\right]
\]

\[
+ \frac{A_r V_{T_i}^2}{2 \omega_i^2 \Omega} \left(\frac{s (k^2 - K_i^2)}{s^2 + z^2}\right)
\]

\[
- \frac{A_r V_{T_i}^2}{2 \omega_i^2 \Omega} \left(\frac{s (k^2 - K_i^2)}{s^2 + z^2}\right)
\]

\[
\Omega \omega_i^2 (s^2 + z^2)^2
\]

Eqs (4) and (5) reduce to Eq. (4) of Gupta and Singh\(^9\) if we put \(k = 0\) and \(A_r = 0\). The parameters \(\epsilon_1\) and \(\epsilon_2\) are dependent on parallel electrostatic field which can be easily computed for any region of the magnetosphere and their effects on propagating whistler wave can be studied. The real part of refractive index is written as

\[
\mu^2 = \frac{\delta^2 s}{\Omega (s^2 + z^2)} \left[1 + \frac{V_{T_i}^2 (k^2 - K_i^2) s^2 + 6 k K_i s z V_{T_i}^2}{2 \omega_i^2 (s^2 + z^2)^2}\right]
\]

\[
- \frac{A_r V_{T_i}^2}{2 \omega_i^2 \Omega} \left(\frac{(k^2 - K_i^2) s + 4 k K_i z}{(s^2 + z^2)}\right)
\]

In general, the presence of magnetic field makes the magnetospheric plasma anisotropic and wave normal subtends an angle with the ray direction which is
known to arise from the anisotropy of the plasma. In an anisotropic medium the wave propagation is schematically shown in Fig. 1. The angle between the whistler wave normal and the whistler ray direction is written as

\[ \tan \alpha = \frac{1}{\mu} \left( \frac{d\mu}{d\theta} \right) \] ...

(7)

Poeverlein demonstrated analytically that the ray direction angle at any point of the refractive index surface is perpendicular to the tangent drawn at that point. The simple geometrical construction of Fig. 1 shows that ray direction makes an angle \( \psi = (\theta - \alpha) \). Differentiating Eq. (6) with respect to \( \theta \) and substituting \( \tan \alpha \) from Eq. (7), we obtain an expression for the ray direction as

\[ \tan \psi = \tan (\theta - \alpha) = \frac{(\cos \theta - 2\Omega \cos \theta - Q \cos \theta)}{1 + (\cos \theta - 2\Omega \cos \theta + Q \sin \theta)} \] ...

(8)

where

\[
Q = \frac{V_i^2 (k^2 - K_i^2)}{2 \omega_i \Omega (\cos \theta - \Omega)} - \frac{A_t V_i^2 (k^2 - K_i^2)}{2 \omega_i \Omega (\cos \theta - \Omega)} \]

In obtaining Eq. (10), we have neglected the Landau term because under certain plasma conditions Landau damping of whistler mode waves does not provide significant contribution and the damping may be caused by some other linear and non-linear processes. In a magnetoplasma the phase velocity of the whistler mode wave is much smaller than the velocity of light \([\omega^2 / k^2] < c^2\] which makes the whistler wave interact with the plasma, resulting into either gain or loss of energy from the ensemble of charge particles. To study the non-convective instability, we have considered \( k \) to be real and \( \omega \) to be complex. Writing \( \omega = \omega_r + j \omega_i \) and separating the real and imaginary parts, we obtain

\[
2(\omega_p^2 + c^2 k^2)\omega_i^3 - \omega_i^2 (2 \omega_H (\omega_p^2 + 3 c^2 k^2)) + \omega_r [2 \omega_H (3 c^2 k^2 + \omega_p^2) + V_i^2 (k^2 - K_i^2) \omega_i (1 + A_t)] - 2 c^2 k^2 \omega_i^3 - A_t V_i^2 \omega_p (k^2 - K_i^2) \omega_i = 0 \] ...

(11)

and

\[
[2 \omega_H (3 c^2 k^2 + \omega_p^2) + 6 \omega_i^2 (c^2 k^2 + \omega_p^2) - 4 \omega_D \omega_i (3 c^2 k^2 + 2 \omega_p^2) + V_i^2 \omega_i^2 (k^2 - K_i^2) (1 + A_t)] \omega_i + 2 k K_i V_i^2 \omega_p^2 \omega_i (1 + A_t) - 2 k K_i V_i^2 \omega_p \omega_H \omega_i = 0 \] ...

(12)

Eq. (11) is third degree equation in \( \omega_i \) and its solution will give three different roots. Assuming that \( \omega_i \) is not much affected by the presence of parallel electrostatic field, we can write

\[
\omega_r = \omega_r(\omega_i) \omega_{r0} = \frac{c^2 k^2 \omega_i}{(c^2 k^2 + \omega_p^2)} \] ...

(13)

Substituting \( \omega_r \) from Eq. (13) into Eq. (12), we obtain an expression for growth/decay rate of propagating whistler mode waves as

\[
\gamma = \frac{\omega_i}{\omega_H} \] ...

(14)
where
\[
\omega_1 = \left( \frac{2K_1 V_{t1}^2 A_{\omega_1}}{c} \right) (\delta^2 + \eta^2)^{\frac{3}{2}}
\]
\[
\omega_h = \frac{2}{\omega_1} \left( \frac{\omega_1^2}{c^2} \right) + V_{t1}^2 (1 + A_T)
\]
\[
\times \left[ K_1^2 - \left( \frac{\omega_1^2}{c^2} \right) \eta^2 (\delta^2 + \eta^2)^{\frac{3}{2}} \right]
\]
and \( \eta = c k / \omega_h \). We find decaying whistler mode for \( \gamma > 0 \), and growing mode for \( \gamma < 0 \). Eq. (14) reveals that the decay or growth of whistler waves is governed by plasma anisotropy and field parameters.

4 Magnetospheric plasma and field parameters

The dipole configuration of the geomagnetic field lines has been assumed. The magnetospheric plasma in the equatorial region is considered to be tenuous and collision free which conforms well with the frozen-in plasma concept. The electron density is written as

\[
N(R, \phi) = 1.99 \times 10^3 \frac{B(R, \phi) \cos^6 \phi}{(1 + 3 \sin^2 \phi)^{3/2}} \text{el./m}^3 \ldots (15)
\]

and the dipolar magnetic field as

\[
B(R, \phi) = \frac{B_0}{L^3} \left( 1 + 3 \sin^2 \phi \right)^{3/2} \cos^6 \phi
\]

where \( \phi \) is the geomagnetic latitude, \( B_0 \) the dipolar magnetic field at ground and \( L \) the McIlwain parameter. The variations of plasma and gyrofrequency along the field lines for particular \( L \) values can be estimated from Eq. (15). The electron temperature in the equatorial plane is given as

\[
T_e = 6.0 \times 10^3 L^{1.9} \ldots (16)
\]

This is an empirical relation based on satellite measurements which implies that the electron temperature along a particular geomagnetic line of force is almost constant. The measured values of electron density and temperature compare fairly well with these models. The GEOS-1 measurement of thermal energy of the plasma around equatorial region of the magnetosphere using mutual impedance probe in the range \( 4 < L < 8 \) is found to be in the range \( 0.2 \text{ eV} < T_e < 115 \text{ eV} \). Further, the electron number density in this region is reported to be less than \( 7.0 \times 10^7 \text{ el./m}^3 \).

Using in situ measurements, the role of parallel electrostatic field has been accounted for in the computation of refractive index surfaces and growth/decay rates of whistler mode waves. The study of whistler wave propagation in hot magnetospheric plasma is important in the case of wave frequencies well below the electron gyrofrequency. In this paper, we have confined ourselves to the study of various forms of refractive index surfaces for whistler mode waves in the frequency range of \( \left( \omega / \omega_h \right) < 0.5 \), which is appropriate for the low latitude whistler wave propagation.

5 Results and discussion

The role of refractive index surfaces in the whistler mode wave propagation in a cold collisionless plasma was first studied by Gendrin and later on discussed by many other workers. The collisional and warm plasmas are known to deform the idealized refractive index surfaces. Using magnetospheric plasma and field parameters as discussed in previous section along with Eq. (3), we have computed the whistler mode refractive index variation with angle of propagation \( \theta \). The plot of \( \mu \) versus \( \theta \) defines the refractive index surface which is symmetrical about the geomagnetic field. The refractive index surfaces for \( \omega = 40 \text{ krad/s} \) whistler wave propagating along the geomagnetic field lines with \( L = 4, 4.5 \) and 5.5 respectively have been computed for isotropic plasma with various values of parallel electrostatic field and normalized \( k = k V_{t1} / \omega_h \) values and are shown in Fig. 2. The solid lines show the refractive index variation with \( \theta \) in the absence of any parallel electric field for various \( k \) values. The dashed lines depict the \((\mu, \theta)\) curves in the presence of parallel electric field of 10, 15 and 20 mV/m respectively for \( k = 0.00 \) at \( \phi = 0 \). It is clear from Fig. 2 that the refractive index surfaces are drastically deformed for certain electric field and changed into closed surface. However, the variation of normalized wave vector \( k \) produces opposite effect on the refractive index surfaces as compared to the parallel electric field. The slope of refractive index shows that the whistler mode energy flow along the magnetic field line is inhibited in the presence of parallel electric fields. It is clear from Fig. 2 that at lower \( L \) values the region of inflection in the refractive index surface is comparatively flat and guidance of whistler mode waves along the magnetic field is more effective. Similar variations of refractive index surfaces in the anisotropic plasma for \( A_T = 0.50 \) at different \( L \) values are shown in Fig. 3. We find from the figure that the refractive index surfaces are deformed in the presence of temperature anisotropy. Thus, temperature anisotropy produces similar effect on refractive index surfaces as collisions and parallel electric fields do.
However, it produces opposite effect as compared to the effect produced by the normalized wave vector \( \tilde{k} \). We have also constructed the varying refractive index surfaces along a fixed geomagnetic field line \( L = 4 \) for various values of \( \phi \) (Fig. 4). The solid lines show the refractive index surfaces in the absence of parallel electric field \( E_0 \) for \( k = 0.00 \). The dashed lines show refractive index surface variation in the presence of \( E_0 = 15 \) mV/m for \( k \to 0.00 \), and the dash-dot-dash lines show the variation of \( E_0 = 0 \) mV/m and \( k = 0.03 \) in the anisotropic plasma. It is found that as the value of magnetic latitude increases, the inflection point of the refractive index surfaces becomes more clear than in the case of \( \phi = 0 \). The effect of parallel electric field and normalized wave vector \( \tilde{k} \) on refractive index surfaces becomes more prominent at higher values of \( \phi \). Using Eqs (8) and (9), we have computed the group velocity direction for different wave normal angles in the absence and presence of parallel electric field for normalized wave vector \( \tilde{k} \) both in isotropic and anisotropic plasma at \( L = 3.0, 3.5 \) and 4.0 respectively. In Fig. 5, we have shown the variations of refractive index surfaces and the group velocity direction with the wave normal angle. The computation has been made for \( \theta \) varying between 0° and 65°. In the magnetosphere the quasi-longitudinal condition holds good for \( \theta = 30^\circ \). It is clearly shown in Fig. 5 that for \( L = 4 \), the angle \( \psi \) increases and then decreases to zero for various \( k \) values. For \( L = 3 \) and 3.5, the angle \( \psi \) steadily increases for \( \theta = 30^\circ \), which is consistent with the quasi-longitudinal approximation, and for \( \theta > 30^\circ \), the ray direction reaches a maximum value and thereafter decreases with increasing wave normal angles. The
trend of the curves for $L = 3.0$ and 3.5 depicts the nature of variations as shown for $L = 4$ but does not satisfy the theoretical requirement of the quasi-longitudinal approximation. It appears that the quasi-longitudinal approximation does not alter these computations appreciably for $\theta > 30^\circ$. Some of the curves in Fig. 5 show a slight increase of group velocity direction, which is same as that shown by the ray direction $\psi$, with the variation of wave normal angle $\theta$ in the presence of parallel electric field of 15 mV/m for $k \rightarrow 0.00$ in isotropic plasma. A similar variation of $\psi$ with $\theta$ is shown in anisotropic plasma at $A_r = 0.50$, and corresponding curves with their notations are shown in Fig. 5. This finding shows that whistler mode energy with an enhanced $k$ value can be focussed along the geomagnetic field whereas the presence of parallel electric field and temperature anisotropy reduce the focussing of whistler wave energy. Thus, in the presence of parallel electric field and temperature anisotropy, the whistler mode energy flux received on the ground is considerably affected and shows the modulations in the trace intensity of whistler sonograms.

We have obtained an expression for growth rate of whistler mode waves. The whistler sonograms at times depict uneven intensity distributions, and under certain conditions they have been found to break and depict the presence of banded whistlers\textsuperscript{23,59}. The cause of banded whistler formation has not been worked out as yet. The banded whistlers are believed to be produced by non-linear effects. We have studied the growth/decay of whistlers using Eq. (14). The variation of growth/decay rate with normalized wave parameter has been shown in Fig. 6 in isotropic and anisotropic magnetospheric plasma for $L = 4$ and $\omega = 40 \text{ krad/s}$. The growth/decay rate is found to increase with increasing values of parallel (antiparallel) electric field. It is clearly shown in Fig. 6 that the presence of temperature anisotropy increases the growth or decay rate of propagating whistler waves as $\eta$ increases, depending on whether the electric field is parallel or antiparallel to the geomagnetic field lines in the region of propagation. The denominator of Eq. (14) shows that the relative magnitudes of two terms would determine the growth or decay of the whistler mode waves which satisfy the inequalities

$$\omega_{Hi}^2 > \frac{1}{2} V_T^2 (K_1^2 - k^2) \left[ 1 + \frac{c^2 k^2}{\omega_B^2} \right] (1 + A_r)$$

and

$$K_1^2 > k^2$$

The exception to these criteria occurs for waves of
large-wave numbers. For $k > K_1$, the whistler mode waves decay for the entire range of frequency, whereas for small values of $K_1$ the waves satisfying the above inequalities grow with time.

6 Conclusion
The electrostatic field parallel to the geomagnetic field controls the dynamics of charged particles and thus affects the propagation of whistler waves and their interaction with the charged particles in the magnetosphere. The focussing or defocussing and growth or decay rate of whistlers may occur simultaneously which may suppress or enhance the recorded intensity in certain parts of the whistler sonograms. Under extreme conditions, the whistler sonograms do not show smooth variations of dispersion which invariably are seen in the form of banded whistlers. The physical mechanism of banded whistlers is not yet well known. However, it is quite likely that the banded whistlers arise due to non-linear effects. The simultaneous measurements of whistler mode wave propagation, electrostatic field and temperature anisotropy would help to visualize the role of electric field, and anisotropy of plasma distributions in the magnetosphere may lead to further diagnostics of the whistlers and their interaction with the magnetospheric charged particles.

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