A simple spectral model of F-region electron densities at equatorial and low latitudes

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Steady state plasma continuity equation for F-region of the earth's ionosphere at low latitudes has been solved by including plasma transport due to electromagnetic (E×B) drift and neutral winds. Electron densities are expressed analytically as a function of dip latitude using Legendre polynomials. Vertical equatorial drift velocity is used as an adjustable parameter so as to fit the ionospheric electron content measured at an equatorial anomaly station Rajkot (lat. 22.3°N, long. 70.7°E, dip lat. 16.5°N). Electron densities are calculated for solar maximum and minimum conditions, three seasons (equinox, summer, winter), and three local times during day (0900, 1300, 1700 hrs). The set of Legendre expansion coefficients have been stored and can be used to generate electron density profiles at any dip latitude. The adjusted values of equatorial drift velocity are compared with the Doppler and incoherent radar measurements, and the results are discussed.

1 Introduction

Some of the major physical processes governing the F-region of the ionosphere of the earth are: ion production due to solar extreme ultraviolet (EUV) radiation, ion-neutral reactions, energy deposited due to auroral particle precipitation, ambipolar diffusion along geomagnetic field lines, bulk transport of plasma along field lines due to neutral winds, transport of plasma perpendicular to field lines due to electromagnetic (E×B) drift. In addition to these processes, geomagnetic storms, plasma instabilities and gravity waves also play important roles in shaping the ionosphere. Plasma density in the F-region is governed not only by the processes occurring within the ionosphere but also through strong coupling with neutral thermosphere and magnetosphere.

In spite of the many complexities involved, there have recently been important advances in the theoretical modelling of the earth's ionosphere. Many regional models of the ionosphere for various latitude regions have also been developed. In particular, the ionosphere at equatorial and low latitudes, which is the subject matter of the present study, has been modelled extensively. Hanson and Moffett solved the steady state continuity equation for the equatorial F-region by including the neutral winds and the electromagnetic drift. Sterling et al. obtained solutions of the time-dependent continuity equation. The electric field responsible for the E×B drift was derived from the dynamo electric potential in that work. Anderson et al. have recently constructed a semi-empirical low latitude ionospheric model with the calculation technique similar to that of the work of Sterling et al. The vertical E×B drift velocities used by them have been taken from the incoherent radar observations reported by Fejer et al. The calculated electron density profiles are then fitted with the help of a function having six parameters. These parameters have been generated by these authors as a function of latitude (every 2° between 24°N and 24°S dip latitude), local time (every half hour), season, and solar activity. Such analytical representations of electron density profiles may be very convenient for some applications. In the present work we describe a simple spectral model of F-region at low latitudes in which the electron densities are represented analytically (using a set of coefficients) as a function of dip latitude. For simplicity we have decided to calculate electron densities using a static rather than a dynamic model. Models based on static calculations have also been developed for mid-latitude F-region by Nisbet and Nisbet and Divany. Thus the steady state single ion (O+) continuity equation for the F-region is solved in the present work by including transport due to diffusion, neutral winds, and vertical E×B drift. The equatorial drift velocity \( w_0 \) is treated as an ad-
justable parameter so as to reproduce the ionospheric electron content (IEC) measured\textsuperscript{16-20} near the peak of equatorial anomaly station Rajkot (lat. 22.3°N, long. 70.7°E, dip lat. 16.5°N) in the Indian sector. The adjusted drift velocities obtained in the present work are compared with the measurements of Fejer et al.\textsuperscript{21} and Namboothiri et al.\textsuperscript{22} Sets of coefficients have been generated at 80 values of an altitude-related parameter for three local times (0900, 1300 and 1700 hrs), three seasons (winter, summer, equinox), and two levels of solar activity (solar maximum and solar minimum). The sets of coefficients along with the computer programme (written in FORTRAN), which calculates the electron density profiles using these coefficients, have been stored.

2 Details of the model

2.1 Model equations

The steady state equation of continuity for the F-region can be written as\textsuperscript{23}

\[
\nabla \cdot \left[ \frac{1}{m_{\text{n}}v_{\text{in}}} \left( Nm_{\text{i}}g' - \nabla' (NkT_{\text{p}}) \right) \right] \\
+ \nabla \cdot (NV') + \nabla \cdot (Nv_{\perp}) = Q_{\text{i}} - L_{\text{i}} \quad \ldots (1)
\]

The first term on L.H.S. of Eq. (1) represents transport due to diffusion, the second term is transport due to neutral winds, and the third term is transport due to electromagnetic drift. Symbol \( V' \) denotes the component parallel to the geomagnetic field. Other quantities appearing in Eq. (1) are:

- \( N \): ion (electron) number density
- \( m_{\text{i}} \): ion mass
- \( g \): acceleration due to gravity
- \( v_{\text{in}} \): ion neutral collision frequency
- \( T_{\text{p}} \): plasma (ion + electron) temperature
- \( k \): Boltzmann constant
- \( U \): meridional neutral wind velocity
- \( v_{\perp} \): \((E \times B)\) drift velocity
- \( Q_{\text{i}} \): ion \((O^{+})\) production rate
- \( L_{\text{i}} \): ion loss rate

For a dipole field it is convenient to transform Eq. (1) using curvilinear field line coordinates \((q, p, \eta)\), where

\[
q = \frac{r_{0}^{2}}{r^{2}} \cos \theta \\
p = \frac{r}{r_{0} \sin \theta} \\
\eta = \phi
\]

Here \((r, \theta, \phi)\) are spherical coordinates of a point and \( r_{0} \) is the radius of the earth. Coordinate system is shown in Fig. 1.

Eq. (1) is transformed to

\[
A \frac{\partial^{2} N}{\partial q^{2}} + B \frac{\partial N}{\partial q} + CN + \frac{w_{0}}{r_{0}} \frac{\partial N}{\partial p} = Q_{\text{i}} - L_{\text{i}} \quad \ldots (3)
\]

Details of transforming the equation of continuity to the field line coordinates are described in the works of Hanson and Moffett\textsuperscript{6} and Sterling et al.\textsuperscript{3} Final expressions for the coefficients are rather lengthy. These are given in Appendix A. Dependent variable \( N \) is next transformed to a new variable \( G \) using

\[
N = G \exp \left[ - \frac{k_{1}}{a T_{\text{e}}} \left( 1 - \frac{r_{0}}{r} \right) \right] \quad \ldots (4)
\]

where \( k_{1} = \frac{m(O)g(r_{0})}{k} \) and \( a = \frac{T_{\text{e}} + T_{\text{i}}}{T_{\text{i}}} \)

and \( m(O) \) is the mass of neutral atomic oxygen and \( g(r_{0}) \) is the value of acceleration due to gravity at the surface. \( T_{\text{e}} \) and \( T_{\text{i}} \) are electron and ion temperatures respectively. The independent variable \( q \) is changed to \( y \) through the transformation

\[
y = \frac{\sinh \Gamma q}{\sinh \Gamma q_{\text{max}}(p)} \quad \ldots (5)
\]

where \( \Gamma \) is a constant (set equal to 12) and \( q_{\text{max}}(p) \) is given by

\[
q_{\text{max}}(p) = \frac{r_{0}^{2} \cos \theta}{r^{2}} \cos \theta
\]

Fig. 1—Dipole field line coordinate system, \((q, \theta, \phi)\) and \((r, \theta, \phi)\) form right-handed coordinate systems; \( \hat{n} \) and \( \hat{\phi} \) point eastward.
\[ q_{\text{max}}(p) = \frac{r_0^2}{r_b^2} \left( 1 - \frac{r_b}{r_0 p} \right)^{1/2} \]  

where

\[ r_b = (r_0 + 160) \text{ km} \]

It may be noted here that \((p, \eta)\) determines a particular field line and \(q\) varies along the field line, \(q = \pm q_{\text{max}}(p)\) at the two ends of the field line where \(|y|\) takes the maximum value 1. Using the transformation Eqs (4) and (5), Eq. (3) is finally written in the form

\[ \frac{w_0}{r_0} \frac{\partial G}{\partial p} \bigg|_{y} = -a \frac{\partial^2 G}{\partial y^2} - b \frac{\partial G}{\partial y} - c G + d' \]  

Transformations \(N \rightarrow G\) and \(q \rightarrow y\) are described in Refs 6 and 7. These are briefly given in Appendix A.

### 2.2 Method of solution

Eq. (7) is solved using a spectral method in which the variable \(G\) and the coefficients appearing in Eq. (7) are expanded in Legendre polynomials (nine in the present work). Spectral method has earlier been used by Whitten et al.\(^{24}\) and Singhal and Whitten\(^{25,26}\) to study the energetics and dynamics in the ionospheres of Venus and Mars. Necessary details of the spectral technique used in the present work are described in Appendix B. The set of coupled equations (B9) were solved approximately by matrix inversion using the approximation

\[ \frac{d g_s}{d p} = \frac{g_s^{i-1} - g_s^{i}}{\Delta p} \]  

Such an approximate procedure was found necessary to use for drift velocities smaller than 15 m/s. For larger drift velocities, equations (B9) were also solved by a Runge-Kutta technique. The two methods yielded electron densities within a few per cent for small value of \(\Delta p = 0.001573\) used in this work. The lower boundary of \(p\) was kept at \(p_{\text{min}} = 1.028316\) and the upper boundary at \(p_{\text{max}} = 1.278423\). Electron densities needed at the lower boundary were obtained assuming photochemical equilibrium. From Eq. (2) it may be seen that these values of \(p\) cover an altitude range from 160 km to 1600 km at the equator and only up to 400 km near dip latitude 24°. The loss term appearing in Eq. (3) is treated implicitly by writing \(L = \mathcal{L} \tilde{N}\) and transferring it on the left hand side of the equation. To reduce the complexity of calculations, we have made the following assumptions in the present work.

1. Electron and ion temperatures are assumed equal (i.e., \(\alpha = 2\)) and of the form \(T_1 = 1400 (1.0 + 0.15 \cos^2 \theta), \theta\) being the colatitude. For solar minimum conditions the ion temperature was increased from 1400 to 1950 K to ensure the numerical stability of the calculations.

2. Neutral constituent number densities and neutral temperature are assumed independent of \(\theta\). The values at equator are used.

3. Steady state conditions are assumed and the equatorial drift velocity \(w_0\) is treated as an adjustable parameter independent of altitude.

### 3 Input data

Neutral number densities for \(N_2, O_2\) and \(O\) and the neutral temperature used in the present study have been taken from the MSIS-86 thermospheric model of Hedin\(^{27}\). Neutral wind velocities have been taken from the empirical global model of Hedin et al.\(^{28}\) Representative altitude versus neutral number density profiles are shown in Fig. 2. Meridional neutral wind velocities are shown in Figs 3(a)-(c) for three latitudes. Positive values refer to north to south winds. Solar EUV fluxes at 37 wavelength intervals have been calculated using the data obtained by spectrophotometers carried aboard the Atmosphere Explorer (AE) satellites and the algorithm of Hinteregger\(^{29}\). Photoionization and photo-absorption cross sections have been taken from Gombosi et al.\(^{30}\) and Torr and Torr!\(^{31}\) For solar zenith angle \((\chi) > 80^\circ\) Chapman grazing incidence function was employed using the expressions given by Smith and Smith\(^{32}\).

![Fig. 2—MSIS-86 neutral densities for latitude = 0°, local time (LT) = 1400 hrs, day count = 289, longitude = 70.7°E, and Ap = 15.](image-url)
For \( x < 80^\circ \) Chapman function was replaced by \( \sec x \), \( x \) was calculated using

\[
\cos x = \sin \lambda \sin \delta 
+ \cos \lambda \cos \delta \cos \left[ 2\pi (HL - 12)/24 \right]
\]

... (9)

where HL is the local time in hours, \( \lambda \) the geographic latitude, and \( \delta \) the solar declination given as

\[
\delta = -0.40915 \cos \left[ 2\pi (D + 8)/365.25 \right]
\]

... (10)

Here \( D \) is the day number. The value of ion neutral collision frequency was taken from Banks and Kockarts\(^\text{23} \). The constant \( n_0 = 1 \times 10^5 \text{ cm}^{-3} \) was added to the atomic oxygen concentration to avoid the problem of instability in the numerical method. Rate constants \( R_1 \) and \( R_2 \) for ion-neutral charge exchange reactions with \( \text{N}_2 \) and \( \text{O}_2 \) were taken as

\[
R_1 = R_0 \left( 300/T_n \right) \text{ cm}^3 \text{s}^{-1}
\]
\[
R_2 = 10 R_0 \left( 300/T_n \right)^{1/2} \text{ cm}^3 \text{s}^{-1}
\]

Here \( T_n \) is the neutral temperature. For conditions of solar minimum \( R_0 = 1.1 \times 10^{-12} \text{ cm}^3 \text{s}^{-1} \) and for solar maximum \( R_0 = 2.7 \times 10^{-12} \text{ cm}^3 \text{s}^{-1} \) were used. These values were obtained by fitting the IEC data at equatorial station and are approximately the same as given by Torr and Torr\(^\text{23} \). Calculations were performed for two conditions of solar activity, solar maximum \( (F_{10.7} \approx 200) \) and solar minimum \( (F_{10.7} = 70) \). Geomagnetic conditions were chosen by setting \( A_p = 15 \). The ionospheric electron content (IEC) data at an equatorial anomaly station Rajkot in the Indian sector were used. These measurements are available during solar minimum (1975-76) from the ATS-6 satellite\(^\text{16-20} \) and during solar maximum (1980) from the ETS-II satellite\(^\text{16} \).

4 Results and discussion

In Fig. 4 the results of a test case for studying the effect of neutral winds and plasma drift on electron densities are shown. A uniform north-south wind of 50 m/s is assumed. Plasma drift speed is taken as 15 m/s. Results are shown for two dip latitudes \( 8^\circ \) and \( 16^\circ \) in the northern and southern hemispheres. The model calculations for the case without any drift or wind are shown by solid lines. The case for equatorial drift velocity included but no winds is shown by dashed lines. Finally by including both the wind and plasma drift we obtain the curves shown by dash-dot-dash (northern) and dash-two dots-dash (southern).
Fig. 4—Electron densities are shown for a test case study at two dip latitudes 8° and 16°. N and S refer to the northern and the southern hemispheres.

Fig. 5—Electron density profiles at equator and 16°S dip latitude.

Fig. 6—Electron density profiles for Rajkot for three seasons and three local times (0900, 1300 and 1700 hrs) during solar maximum.

From Fig. 4 we note that the effect of electromagnetic (E×B) drift is to transport the plasma to higher latitudes. This is the 'fountain effect' in which the ionization is lifted upwards at low latitudes and then diffused along the field lines, envisaged by Martyn to account for the Appleton or equatorial anomaly. Mitra had earlier suggested, although incorrectly, that plasma diffusion along field lines alone is responsible for the equatorial anomaly. It may further be noted that at lower altitudes higher electron densities are obtained in the southern hemisphere due to the transport of plasma by a north-south wind. At higher altitudes, however, the electron densities are lower in the southern hemisphere since the plasma is transported to higher altitudes in the northern hemisphere where losses are less and to lower altitudes in the southern hemisphere where the losses are more. These results are in qualitative agreement with the work of Hanson and Moffet. The differences, particularly at higher altitudes, are mainly due to the limited accuracy of the present calculations, a point which will be discussed in the next paragraph.

In Fig. 5 the results of present model calculations are compared with those obtained with the more accurate time-dependent model of Anderson et al. and with the empirical electron density profiles given by Chiu. At higher dip latitudes the present model overestimates the electron densities around the peak. Furthermore, the present electron densities fall off much faster with increasing altitude as compared to the electron densities calculated by Anderson et al. These differences are in part due to the use of a smaller (nine) number of Legendre polynomials. Increasing the number of Legendre polynomials becomes very time consuming on the computer, apart from posing problems relating to matrix inversion (with nine terms it takes about 30 min on the Mighty Frame II for each run). The calculated electron density profiles at Rajkot are shown at three local times during day for different seasons during the solar maximum and solar minimum conditions (Figs 6 and 7 respectively). For solar minimum conditions, the electron density calculations could not be carried out for the complete range of the values of the parameter p [defined in Eq. (2)]. For values of $p \geq 1.18$ the numerical method tends to become unstable for this case. The altitude range covered for solar minimum is therefore much lower as compared to calculations for solar maxi-
In Figs 8-10 the IEC values calculated in the present work are compared with the experimental data available at low latitudes. The calculated IEC values for solar maximum are for the year 1980 ($F_{10.7} = 200$). The data of Rastogi et al.\textsuperscript{37} (shown by small symbols; lower panels) are for the period 1967-69 ($F_{10.7} = 148$). The calculated values are much higher in comparison with the observed data. Differences are more pronounced during winter and equinox since the observed $E\times B$ drift velocities (Ref. 21) for these seasons are larger than those for the summer months. The calculated and the observed values of IEC for solar minimum ($F_{10.7} = 72$) are in reasonable agreement. It may however be noted that the values of calculated IEC, in general, tend to be lower than those of the observed data. This is because the values of IEC for solar minimum have been obtained by numerical integrations of calculated electron densities up to altitude of nearly 700 km only. Furthermore, the values of calculated IEC fall off much faster with increasing dip latitude as compared to the observed values. The reason for this, as mentioned earlier, is that for the solar minimum case the range of $p$ values was limited to $p < 1.18$ due to problems in the numerical method. At higher dip latitudes the electron densities have been integrated up to altitude of 300 km only. The data points at dip latitude 16.5°N (Rajkot) (Ref. 16) are for the period 1975-76 ($F_{10.7} = 72$) and at dip latitude 11.7°N (Bombay) (Ref. 38) are for the period Jan.-Dec. 1984 ($F_{10.7} = 100$). A closer examination of Figs 8-10 for the solar min-
imum case reveals that for LT=1300 hrs observed equatorial anomaly crest lies at higher dip latitude in comparison with the model prediction. This may suggest that the $E \times B$ drift velocities during the period 1964-66 might have been larger than those obtained in the present work for the period 1975-76.

Finally in Figs 11-13, the peak electron densities ($N_m$) calculated in the present work are compared with the experimental data available at low latitudes. For solar minimum the agreement between the model calculations and the observed data is in general reasonable. The calculated $N_m$ values at LT=1700 hrs tend to be somewhat lower than the observed values. For solar maximum, model calculations predict higher $N_m$ values near the equatorial anomaly crest in comparison with the observed values of Rastogi et al. The reason for this is in part due to the fact that the data of Rastogi et al. refer to the period 1964-66 ($F_{10,7} \approx 148$) and the model calculations are for the period 1980 ($F_{10,7} \approx 200$). It may however be remarked that the observed values of $N_m$ for the year 1980 are same as for the period 1964-66 even though the solar maximum of 1980 was much stronger than the one during 1967-69. In Fig. 11 we have also shown the observed $N_m$ values for the month of March 1958 taken from the work of Rao and Malhotra. The agreement between the model calculations and these observations is somewhat better. This may be expected since the solar activity for this period was very strong ($F_{10,7} \approx 250$). F-region electron densities are very sensitive to the $E \times B$ drift velocities and neutral winds. These quantities are known to undergo large day-to-day variations. The comparison between model calculations of the present work and the observations which refer to different periods should therefore be viewed with caution.

As noted earlier the present electron density calculations are based on a static model in which the equatorial drift velocity $w_0$ is adjusted so as to fit the observed IEC values at Rajkot. The adjusted values of $w_0$ are given in Table 1, from which it may be noted that the adjusted drift velocities are in reasonable agreement with the measurements of Fejer et al. which correspond to the conditions of solar maximum. However, for solar
Table 1—Vertical drift velocities in the F-region

<table>
<thead>
<tr>
<th>Season</th>
<th>Local time (hrs)</th>
<th>Solar maximum</th>
<th>Solar minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Adjusted</td>
<td>Measured*</td>
<td>Adjusted</td>
</tr>
<tr>
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<td>11</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>1300</td>
<td>17</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>1700</td>
<td>27</td>
<td>15</td>
</tr>
<tr>
<td>Winter</td>
<td>0900</td>
<td>9</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>1300</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>1700</td>
<td>22</td>
<td>14</td>
</tr>
<tr>
<td>Summer</td>
<td>0900</td>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>1300</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>1700</td>
<td>14</td>
<td>11</td>
</tr>
</tbody>
</table>

*Fejer et al.21; **Namboothiri et al.22

minimum case the adjusted drift velocities are in general much higher than the values measured by Namboothiri et al.22 which also refer to solar minimum. These differences are probably due to the technique of measurement as has been pointed out by Fejer et al.21 Namboothiri et al.22 have used the HF Doppler radar for measuring the vertical drift velocities. Fejer et al.21 have used the incoherent scatter radar technique. It thus appears that the drift velocities measured by the HF Doppler radar technique are largely underestimated specially during equinox and summer seasons.
5 Conclusions
A model of F-region electron densities at equatorial and low latitudes has been constructed using a spectral technique. The calculated electron density profiles are in reasonable agreement with other model calculations and data. The adjusted values of equatorial drift velocity are in good agreement with the values measured by Fejer et al.21 during solax maximum. For solar minimum the measurements of drift velocities by Namboothiri et al.22 are much lower than our adjusted values. The stored set of Legendre expansion coefficients along with the programme to generate electron density profiles at any dip latitude can be useful in many applications for the Indian sector. Further improvements in the model should, however, be made by incorporating more realistic meridional neutral wind velocities, an improved matrix inversion technique, and a more stable numerical method to solve coupled first order differential equations. Further modelling efforts should also take into account the variations of ion-neutral reaction rate coefficients and the plasma temperature with local time, season, and solar activity.

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References
Appendix A

Coefficients appearing in the equation of continuity

The three terms on the L.H.S of Eq. (1) represent plasma transport due to diffusion, neutral winds, and electromagnetic drift respectively. Transformation of Eq. (1) using field line coordinates assuming a dipole field has been carried out by Hanson and Moffett and Sterling et al. Final expressions are rather lengthy. We repeat them here for the sake of completeness.

Diffusion term

\[ \nabla \cdot \left[ \frac{1}{m_i v_{in}} [Nm_i g - \nabla (NkT_i^2)] \right] = A_d \frac{\partial^2 N}{\partial q^2} + B_d \frac{\partial N}{\partial q} + C_d N \]

\[ A_d = -\sigma_0^4 \frac{D_a}{r^6} \]

\[ B_d = \frac{2 r_0^2 \cos \theta}{a H} \frac{D_a}{r^3} + \frac{2 r_0^2 \cos \theta}{r^3} \frac{\partial D_a}{\partial r} + \frac{r_0^2 \sin \theta}{r^2} \frac{\partial D_a}{\partial \theta} \]

\[ C_d = \frac{4 \cos^2 \theta}{a \sigma} \frac{\partial D_a}{\partial r} \frac{D_a}{r^2} \]

Neutral winds

\[ \nabla \cdot (NU) = B_n \frac{\partial N}{\partial q} + C_n N \]

\[ B_n = -\frac{a^2 \sin \theta}{r^3} U \]

\[ C_n = -\frac{a^2 \sin \theta}{r^3} \frac{\partial U}{\partial q} + \frac{U \sin \theta}{r^2} (15 \cos^3 \theta + 13 \cos \theta) \]

Here \( U \) is meridional neutral wind (+ve sense is from north to south).

(E x B) drift

\[ \nabla \cdot (NV_x) = C_{em} N \]

\[ C_{em} = \frac{1}{r_0} \frac{\partial w_0}{\partial p} + \frac{4 w_0}{r \sigma^2} \]

\[ \times (6 \cos^6 \theta - 3 \cos^4 \theta - 4 \cos^2 \theta + 1) \]

\( w_0 \) is +ve upwards.

Finally

\[ A = A_d \]

\[ B = B_d + B_n \]

\[ C = C_d + C_n + C_{em} \]

Transformation \( N \rightarrow G \)

\[ N = G \exp \left[ -\frac{K_1}{a T_i} \left( 1 - \frac{r_0}{r} \right) \right] \]
where

\[ K_1 = \frac{m(O) g(r_0) r_0}{k} \]

Letting

\[ f = - \frac{K_1}{\alpha T_1} \left( 1 - \frac{r_0}{r} \right) \]

\[ \frac{\partial N}{\partial q} = \left[ \frac{\partial G}{\partial q} + \frac{G}{\partial q} \right] \exp(f) \]

\[ \frac{\partial^2 N}{\partial q^2} = \left[ \frac{\partial^2 G}{\partial q^2} + 2 \frac{\partial G}{\partial q} \frac{\partial f}{\partial q} + G \left( \frac{\partial f}{\partial q} \right)^2 + G \frac{\partial^2 f}{\partial q^2} \right] \exp(f) \]

\[ \frac{\partial N}{\partial p} = \left[ \frac{\partial G}{\partial p} + \frac{G}{\partial p} \right] \exp(f) \]

\[ \frac{\partial r}{\partial q} = \frac{r_0 \sin^4 \theta}{\alpha} \]

\[ \frac{\partial r}{\partial p} = -2r^3 \cos \theta \]

\[ \frac{\partial^2 r}{\partial q^2} = \left[ 16qpr \frac{\partial r}{\partial q} + 12pq^2 r^2 \left( \frac{\partial r}{\partial q} \right)^2 + 2pr^2 \right] \]

\[ \frac{\partial \theta}{\partial q} = -2pq \sin^3 \theta \]

\[ \frac{\partial \theta}{\partial p} = -2q^2 \sin \theta \]

\[ \frac{\partial f}{\partial q} = \frac{2K_1 r \cos \theta}{\alpha r_0 T_1 \sigma} - \frac{K_1 r^2}{\alpha r_0} \left( \frac{1 - r_0}{r} \right) \frac{1}{T_1} \frac{\partial T_1}{\partial \theta} \sin \theta \]

\[ \frac{\partial f}{\partial p} = -\frac{K_1 \sin^3 \theta r_0}{\alpha T_1 \sigma r^2} \left[ \sin \theta + \left( \frac{r}{r_0} - 1 \right) \frac{1}{T_1} \frac{\partial T_1}{\partial \theta} 2 \cos \theta \right] \]

Transformation \( q \rightarrow y \)

\[ y = \frac{\sinh \Gamma q}{\sinh \Gamma q_{\max}(p)} \]

\[ \frac{\partial G}{\partial q} = L \frac{\partial G}{\partial y} \]

\[ \frac{\partial^2 G}{\partial q^2} = L^2 \frac{\partial^2 G}{\partial y^2} + \Gamma^2 y \frac{\partial G}{\partial y} \]

\[ \frac{\partial G}{\partial p} = \frac{\partial G}{\partial y} \frac{\partial y}{\partial p} + \frac{\partial G}{\partial y} \frac{\partial y}{\partial p} \frac{\partial p}{\partial q} \]

\[ L = \Gamma \frac{\cosh \Gamma q}{\sinh \Gamma q_{\max}} \]

\[ \frac{\partial y}{\partial q} = -y \Gamma \cosh \Gamma q_{\max} r_0^3 \]

collecting terms, we obtain

\[ a' = A L^2 \]

\[ b' = 2A L \frac{\partial f}{\partial q} + B L + A \Gamma^2 y + \frac{w_0}{r_0} \frac{\partial y}{\partial p} \]

\[ c' = A \left( \frac{\partial f}{\partial q} \right)^2 + A \frac{\partial^2 f}{\partial q^2} + B \frac{\partial f}{\partial q} + \frac{w_0}{r_0} \frac{\partial f}{\partial q} + C \]

\[ d' = (Q_1 - L_4) \exp(-f) \]

Inverse transformations of Eq. (2)

Eq. (2) expresses \( q, p, \eta \) in terms of \( r, \theta, \phi \). Inverse transformation, i.e. \( r, \theta, \phi \) can be obtained in terms of \( q, p, \eta \) using

\[ r = \frac{c + 1}{2} \sqrt{c^2 - 4d} \]

\[ c = \sqrt{\beta} \]

\[ \beta = \frac{r_0^2}{q} \left[ \left( \frac{1}{4q^2 p^4 + 64} \right)^{1/2} + \frac{1}{2} \frac{1}{p^2} \right]^{1/3} \]

\[ \beta = \left( \frac{1}{4q^2 p^4 + 64} \right)^{1/2} - \frac{1}{2} \frac{1}{p^2} \left[ \frac{1}{2} \frac{1}{p^2} \right]^{1/3} \]

\[ d = \frac{1}{2} \left( \beta - \frac{r_0^3}{p^2} \right) \]

\[ \theta = \cos^{-1} \left( \frac{qr_0^2}{r_0^2} \right) \]

\[ \phi = \eta \]

Appendix B

Spectral method

Eq. (7) given in section 2.1 in the main text is solved by a spectral technique in which various quantities are expanded in a series of Legendre polynomials. First we write \( G \) as

\[ G = \sum_s g_s(p)(1 - y^2)^s P_s(y), \quad s = 0.1 \quad \ldots \quad (B1) \]
Substituting (B1) in Eq. (7) and making necessary simplifications, we obtain

\[
\frac{w_0}{r_0} \sum_i \frac{d g_i}{d p} P_i = \sum_i g_i P_i \frac{a'}{(1-y^2)} \{l(l+1) + 2s\}
+ \sum_i g_i P_i \left\{ \frac{2ysb'}{(1-y^2)} - \frac{4y^2 s(s-1)a'}{(1-y^2)^2} - b' \right\}
+ \sum_i g_i P_i \left\{ \frac{(4s-2)ya'}{(1-y^2)} - b' \right\} + \frac{d'}{(1-y^2)} \ldots (B2)
\]

Now we make further expansions as follows,

\[
\frac{(4s-2)ya'}{(1-y^2)} - b' = \sum_i e_i P_i \ldots (B3)
\]

\[
\frac{2ysb'}{(1-y^2)} - \frac{4y^2 s(s-1)a'}{(1-y^2)^2} - c' = \sum_i h_i P_i \ldots (B4)
\]

\[
\frac{a'}{(1-y^2)} = \sum_i f_i P_i \ldots (B5)
\]

\[
\frac{d'}{(1-y^2)^2} = \sum_i d_i P_i \ldots (B6)
\]

Now we substitute Eqs B(3)-B(6) in Eq. (B2) and making use of the relations

\[
P_i P_m = \sum_i (2l+1) \begin{pmatrix} l & m \\ 0 & 2 \end{pmatrix}^2 P_i \ldots (B7)
\]

\[
\frac{d P_i}{dy} = \sum_{n=0}^{N} (2l-1-4n) P_{i-l-2n} \ldots (B8)
\]

We obtain a set of first order coupled differential equations

\[
\frac{w_0}{r_0} \sum_i \frac{d g_i}{d p} P_i = \sum_i A_{kl} g_i + d_k, \quad k = 0, 1, 2, \ldots \ldots (B9)
\]

The matrix \( A_{kl} \) is given by

\[
A_{kl} = \sum_{m} e_m (2l-1-4n)(2k+1) \begin{pmatrix} l & 1-2n & m & k \\ 0 & 0 & 0 \end{pmatrix}^2
+ \sum_{m} h_m (2k+1) \begin{pmatrix} l & m & k \\ 0 & 0 & 0 \end{pmatrix}^2
+ \sum_{m} f_m \frac{l(l+1)+2s}{(2k+1)} \begin{pmatrix} l & m & k \\ 0 & 0 & 0 \end{pmatrix}^2 \ldots (B10)
\]

In Eqs (B7) and (B10) we have used Wigner coefficients \( \begin{pmatrix} l & m & n \end{pmatrix}^2 \) (e.g. Edmonds\(^40\)) to combine \( P_i \) and \( P_m \).