Study of undertow oscillations using an analytical model and some numerical simulations

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Undertow is one of the most common phenomena governing the ocean beaches. This study investigates the effects of hydrodynamic parameters and bed slope on the undertow oscillations by means of an analytical model and some numerical simulations. Model predictions are compared with numerical simulation results of waves and currents velocities throughout the entire undertow system and show some reasonable agreement. The results indicate that undertow velocity increases with parametric forcing intensity (γ). So that if γ violates a specific limit, the velocity of undertow will decrease. The numerical results of this study also show that undertow velocities are larger over steeper beaches than on gently sloping beaches.

Keywords: Undertow, Surf zone, Hydrodynamic parameters, Parametric forcing intensity

Introduction

Undertows are the offshore-directed mean currents in the surf zone and the action of these currents is considered as a significant factor in the water circulation of beach areas1. Undertows are driven by radiation stress and setup gradients caused by wave breaking and they are fed by the water volume brought in towards the beach by the incident waves2-6. Furthermore, undertow currents which were considered to have a controlling effect on shore-face erosion of a beach are responsible for the formation of the middle and offshore bars7-10. Therefore, in order to investigate the cross-shore sediment transport in the surf zone, it is necessary to study the cross-shore distribution of the undertow current velocity11.

There have been numerous theoretical and laboratory studies of undertow currents in the vicinity of the surf zone12-15. The undertow profile and the horizontal and vertical distributions of undertow velocity have also been studied by Hansen and Svendsen16, Dally and Dean17, Svendsen, Schäffer and Hansen18, Rattanapitikon and Shibayama19, Neshaei and Mehrad20. Also the previous field experiments have showed a tendency towards the over prediction of the undertow velocities21-23. Abedimahzoon and Neshaei attempted to calculate undertow velocity, based on experimental data in the presence of partially standing waves. The findings of their study demonstrated that the existence of standing waves caused a decrease in the magnitude of the mean flow undertow, and made a change in the distribution of the current velocity across the width of the surf zones. The occurrence of the undertow and rip currents is observed as the waves approach the beach at near normal incidence angle. The case of normal incidence of waves for both the analytical model and numerical simulation is considered in this study. The purpose of this study is to investigate the effects of hydrodynamic parameters and bed slope on the undertow current velocity oscillations by means of an analytical model and some numerical simulations using MIKE 21/3 numerical model. The study is structured as follows: In section 2, the undertow as a kind of return flows that generated by wave breaking is investigated using an analytical model. In section 3, the undertow oscillations are studied using some numerical simulations. In section 4, a summary of the results of the models are compared with the results of other studies. Finally, the conclusions are given in the fifth section.
Materials and Methods

Description of the Analytical Model

The generation of undertow currents owes to waves propagating towards shores, then shoaling and breaking at alongshore topography. Shoaling and wave breaking in the near shore zone occurs by the reduction in depth as waves approach the shore.

The dominated equation to explain the undertow as a kind of near shore flow by means of the mean crossshore current velocity ($u$) may be given as follows:

$$\frac{\partial u}{\partial t} = -g \sin \alpha (\bar{h} - h_0) + F \tag{1}$$

$$\frac{\partial \bar{h}}{\partial t} = -\bar{u} \frac{\partial \bar{h}}{\partial x} + L \eta \tag{2}$$

where $\bar{h}$ is the mean sea water depth of shore zone, $\alpha$ is a beach slope, $h_0$ is the water depth in offshore zone, $g$ is gravitational acceleration, $F$ is the resultant of the forces from, $S$ the force from short wave, $B$ the bed friction and $L$ which is related to horizontal mixing. $L \eta$ is the sea water level change, related to the radiation stress of waves incoming to the shore.

As waves approach the surf zone and reach the shallow region, they break when hitting the bed bars. In fact wave will face the depth change along the x axis, by the use of $H = \gamma (h + \eta)$, can be given as follows:

$$\frac{\partial \bar{h}}{\partial x} = \frac{\partial \bar{h}}{\partial \gamma} \frac{\partial \gamma}{\partial x} = -\frac{1}{\gamma} \sin \alpha$$

where $\gamma$ is a breaker criterion (parametric forcing intensity) and $\eta$ is sea level change. Now, the result of equation (2) is as follows:

$$\frac{\partial \bar{h}}{\partial t} = -\frac{\bar{u}}{\gamma} \sin \alpha + L \eta \tag{3}$$

To consider the effect of hydrodynamic forces on the dynamics of the coastal flows we differentiate of the momentum equation (1) and substitute in equation (3), and then the result is as follows:

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{g\bar{u}}{h_0}\right) \sin^2 \alpha - \left(\frac{g}{h_0}\right) L \eta \sin \alpha + \frac{\partial F}{\partial t} \tag{4}$$

The friction force (due to the friction and mixing) may be given by:

$$F = -k \bar{u} \tag{5}$$

where $k$ is the friction coefficient. Equation (4) can be rewritten in the form of an ordinary differential equation as follows:

$$\frac{d^2 \bar{u}}{dt^2} + k \frac{d\bar{u}}{dt} + \left(\frac{g\bar{u}}{h_0}\right) \sin^2 \alpha = \left(\frac{g}{h_0}\right) L \eta \sin \alpha \tag{6}$$

This equation is a non-homogenous, equivalent to the classic problem of a mass on a spring with damping coefficient ($k$) (here damping is mainly due to bottom friction). The equation has three forms of solutions depending on the relative magnitudes of $k^2$ and $(g/y_0)sin^2 \alpha$. For $k^2 \geq 4(g/y_0)sin^2 \alpha$ the current is subjected to a strong retarding action and proceeds directly to an equilibrium state proportional to the forcing $L \eta$. For $k^2 > 4(g/y_0)sin^2 \alpha$ the solution exhibits a damped oscillatory behavior about the equilibrium position with only small slope and small bottom friction coefficient. For $k^2 < 4(g/y_0)sin^2 \alpha$ the solution displays a damped oscillatory behavior.

In most cases, the damped oscillatory solution behavior equation is indicated as follows:

$$\bar{u} = Ae^{\frac{1}{2}k t} \cos(\omega t - B) - L \eta \gamma \sin \alpha \tag{7}$$

$$\omega = \left[4 \left(\frac{g}{h_0}\right) \sin^2 \alpha - k^2 \right]^{1/2}$$

$A$ and $B$ are fixed coefficients determined by the initial condition. By means of the initial condition the result is as follows:

$$\frac{d\bar{u}(0)}{dt} = 0 \text{ and } \bar{u}(0) = 0$$

yields

$$\begin{cases} A = L \eta \gamma \left(k^2 + 4 \omega \right)^{1/2} / 2 \omega \sin \alpha \\ B = -\tan^{-1} \left( \frac{1/2 k}{\omega} \right) \end{cases}$$

So $\bar{u}$ is given as follows:

$$\bar{u}_{undertow} = -\frac{L \eta \gamma}{\sin \alpha} \left[1 - \left(k^2 + 4 \omega \right)^{1/2} e^{2 \omega / \omega} \cos(\omega t - B) / 2 \omega \right] \tag{8}$$

Therefore, the undertow velocity is related to parameters such as beach slope, $\gamma$, bed friction coefficient, the rate of sea water level change, and other parameters. Also, the period of the related
undertow can be calculated by the following equations:

\[
\omega^2 = \left( \frac{g}{\gamma h_0} \right) \sin^2 \alpha - \frac{1}{4} k^2
\]

\[
T = 2\pi \left[ \left( \frac{g}{\gamma h_0} \right) \sin^2 \alpha - \frac{1}{4} k^2 \right]^{1/2}
\]

(Result of the Analytical Model)

Figure 1 shows the pulsating movement pattern of the undertow based on velocity in the different parametric forcing intensity \( \gamma \). This figure indicates that by the increase of \( \gamma \), the amplitude of the oscillations of the undertow velocity increases, and the period of the oscillation also increases. In other words, the velocity and durability of the undertow change, based on the dominated hydrodynamic conditions of the seashore area. But if \( \gamma \) violates a specific limit, the velocity will decrease, and the frequency of the oscillations (number of pulses) will smoothly be reduced.

According to Figure 1, if \( \gamma \) reaches 0.8, then \( k^2 < 4(\gamma h_0)\sin^2 \alpha \) and the velocity change shows a damped oscillation behavior in balance state.

The undertow velocity with the same hydrodynamic conditions (with the same \( \gamma \)) for beaches with different slope also indicates that the durability of undertow pulses decreases with increasing beach slope, as the slope change will put dramatic effect on the oscillations of sea level near shore.

As a result, undertow velocity is also under the influence of beach slope. The undertow velocity change shows that the undertow velocity in beaches with larger slopes say \( \sin \alpha = 0.01 \) will be larger than those with gentler slope \( \sin \alpha = 0.0075 \) with fixed \( \gamma \).

This is so for the beaches with different water levels \( (L_\eta) \) (Fig. 2).

Figure 3 shows the period of oscillations of the current versus \( H/h \) for beaches with different slopes. This figure indicates that if the beaches with different slopes have fixed Hydrodynamic conditions \( (\gamma=\text{constant}) \), then the larger the beach slope the less is the undertow period, in other words durability of each pulse will be reduced.

In order to get some more insight to these behaviors we did some numerical simulations which are presented below.

Numerical Model

The simulations were performed with MIKE 21/3 Coupled FM Model, a numerical model developed by DHI water and environment. This dynamical modeling system was adopted for application of currents within coastal environments and morphological assessment of the nearshore zone due
to the various hydrodynamic conditions. Therefore, mutual interaction between waves and currents was investigated using a dynamic coupling between the Hydrodynamic module, the Spectral Wave module and the Sand Transport module.24

(a) Hydrodynamic Module (HD)

In this research, the two-dimensional model in HD module was used to investigate wave-induced currents. The model is based on the equations of shallow water in which the depth-averaged velocities of Navier-Stokes equations are integrated in an incompressible fluid.

The depth integrated continuity equation and the horizontal momentum equations over the depth \( h=\eta+d \) are written as:

\[
\frac{\partial h}{\partial t} + \frac{\partial h \vec{u}}{\partial x} + \frac{\partial h \vec{v}}{\partial y} = \frac{h s}{(10)}
\]

\[
\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{u}^2}{\partial x} + \frac{\partial \vec{u} \vec{v}}{\partial y} = gh \frac{\partial \eta}{\partial x} - \frac{h \rho_a}{\rho_b} \frac{\partial \rho}{\partial x} + \frac{gh^2}{2 \rho_b} \frac{\partial \rho}{\partial x} + \frac{\tau_{sx}}{\rho_b} + \frac{\tau_{sy}}{\rho_b} \left( \frac{\partial \vec{u}}{\partial x} + \frac{\partial \vec{v}}{\partial y} \right) + \frac{\partial h T_x}{\partial x} + \frac{\partial h T_y}{\partial y} + h u S \tag{11}
\]

\[
\frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{u} \vec{v}}{\partial x} + \frac{\partial \vec{v}^2}{\partial y} = gh \frac{\partial \eta}{\partial y} - \frac{h \rho_a}{\rho_b} \frac{\partial \rho}{\partial y} + \frac{gh^2}{2 \rho_b} \frac{\partial \rho}{\partial y} + \frac{\tau_{sy}}{\rho_b} + \frac{\tau_{sx}}{\rho_b} \left( \frac{\partial \vec{u}}{\partial x} + \frac{\partial \vec{v}}{\partial y} \right) + \frac{\partial h T_y}{\partial y} + h v S \tag{12}
\]

\[u \text{ and } v \text{ are the depth-averaged velocities defined by:}
\]

\[h \vec{u} = \int_{-d}^{0} u dz, \quad h \vec{v} = \int_{-d}^{0} v dz \tag{13}\]

The lateral stresses \( T_{ij} \) consists of viscous friction, turbulent friction, and differential advection which is estimated using an eddy viscosity formulation, based on of the depth average velocity gradients.

\[
T_{sx} = 2A \frac{\partial \vec{u}}{\partial x}, \quad T_{sy} = A \left( \frac{\partial \vec{u}}{\partial y} + \frac{\partial \vec{v}}{\partial x} \right), \quad T_{yy} = 2A \frac{\partial \vec{v}}{\partial y} \tag{14}
\]

In these relations, \( h(x,y,t) \) is the depth of water; \( \eta(x,y,t) \) is sea surface elevation; \( g \) is gravity acceleration; \( p_a \) \( x,y,t \) is atmosphere pressure; \( \rho_w \) is the water density; \( s_{xx}, s_{xy}, s_{yy} \) and \( s_{yy} \) are components of the radiation stress tensor and \( (\tau_{sx}, \tau_{sy}) \) and \( (\tau_{bx}, \tau_{by}) \) are the \( x \) and \( y \) components of the surface wind and bottom stresses.

(b) Sediment Transport Module (ST)

The method which is used in ST module for calculation of sediment transport is based on the model of combined wave and current and to do this the sediment transport tables for interpolation was used.

(c) Spectral Wave Module (SW)

In this module the spectral wave formulation was used which is based on the conservation equation of wave energy density, designed by Komen and Young and is defined as:

\[
\frac{\partial N}{\partial t} + \nabla \cdot (N \vec{v}) = \frac{s}{\sigma} \tag{15}
\]

In which \( N(x,y,\sigma,t) \) is the spectra density, \( \vec{v}=(c_x, c_y, c_{\sigma}, c_{\phi}) \) is the propagation velocity of a wave group and \( S \) is the source term for the energy balance equation. The propagation velocities of a wave groups are achieved from below relations:

\[
(c_x, c_y) = \frac{d \xi}{dt} = \vec{c}_g + \vec{U} \tag{16}
\]

\[
c_\sigma = \frac{d \sigma}{dt} = \frac{\vec{\sigma} \cdot \vec{U}}{\vec{\sigma} \cdot \vec{U}} \tag{17}
\]

\[
c_\theta = \frac{d \theta}{dt} = -1 \left[ \frac{\vec{\sigma} \cdot \vec{d}}{\vec{d} \cdot \vec{m}} + k \frac{\vec{\sigma} \cdot \vec{U}}{\vec{\sigma} \cdot \vec{U}} \frac{\vec{m}}{\vec{m}} \right] \tag{18}
\]

Where \( s \) is the space coordinate in wave direction \( \theta \), and \( m \) is a co-ordinate perpendicular to \( s \).

Model Domain

In this study, a plan beach with the dimensions 950 m in width (cross-shore) and 1500 m in length (longshore) was considered. The \( x \) axis is normal to the shore and \( y \) along the shore. The area under study was modeled in an unstructured mesh which includes 1352 computation nodes and 2491 elements (Fig. 4). The distances among the regular grids in this model are typically about 5 meters.
Description of the Numerical Model

In this model, oscillations of undertow were studied for 2 hours, and to satisfy Courant Friedrich Levy condition, time step was considered one second. In the Hydrodynamic module, the kind of bed resistance with the fixed manning number which was chosen as a fixed number of 32 m^{1/3}/s and the horizontal eddy viscosity which was chosen with fixed value of 0.28 m^{2}/s. In addition, in the Sand Transport module the wave and current effect are considered at the same time. In this module, the particle diameter of used sediment is fixed as 0.2 mm. In Spectral Wave module, the full spectral formation was used in the formulation of the stationary formula. In this model, also the effect of wind was considered in the lowest order, while the bottom friction coefficient was taken as 0.004 for all cases.

Result of Numerical Model

The results of model for different values of $\gamma$ in the beaches with different slopes for two-hour of integrations are presented. The existence of undertows in beach is shown in figure 5, that indicating the flow pattern consisting of undertows with different magnitudes and directions.

To analyze the results closely, first the position of the undertows by means of current pattern in the model was determined, and then an undertow channel as a sample was chosen (Fig. 6).

The oscillations of cross-shore current (undertow) velocity ($u$) in time series indicate that the undertow velocity increases with $\gamma$ (Fig. 7). But if $\gamma$ violates the specific limit ($\gamma = 0.8$), the undertow velocity will be reduced, a behavior very similar to the result of the analytical model (Fig. 1).
Fig. 7 — Time series related to the oscillations of undertow velocity for different \( \gamma \) in the studied area.

Also numerical model in a shore with the slope 0.01 was compared with another beach with a different slope 0.0075 with the same hydrodynamic conditions (\( \gamma = 0.75 \)) by means of time series in the studied area (Fig. 8).

The results of time series indicate that the undertow velocity is more intense in the beach with larger slope, and the number of oscillations also increases in the beaches with larger slopes, but the period of each pulse in the beaches with larger slopes is reduced. This is related to the variations of sea surface elevation with time, so that the oscillations of the sea surface elevations are more intense with time in the beaches with larger slopes (Fig. 9).

Discussion

The comparison of time series of the cross-shore current (undertow) velocity \( (u) \) by means of both an analytical model and a numerical simulation show reasonable agreement. It should be pointed out that the results of the numerical model are compared with the results of analytical model after 45 minutes or "warm up time". In both models as the value of \( \gamma \) increases up to 0.8 the current speed increases, and beyond that then the speed of the undertow decreases and even the pattern of current speed changes (Figs. 1, 7).

Simultaneous results of analytical and numerical solutions are shown in figure 10 that shows close similarity.

Also the results of research in both models show that in a beach with the specific slope, the period of
each undertow pulse or the durability of each pulse increase by the increase in $\gamma$ (Fig. 3).

In comparison of Figs 2 and 8 it can be stated that the increase of beach slope causes the increase of undertow current speed and the number of current pulses. Also, the results of both analytical and numerical models are the same for the beach slope changes. But the numerical model results can clearly show that the beach slope changes can affect the oscillations of the sea surface elevation and this can lead to changes in undertow velocity with time. The similar results about the increase of undertow velocity with the increase of beach slope have been demonstrated by other researchers through field studies.

Investigating the pattern of cross-shore current velocity in figure 5 shows that the instability of current decreases in the offshore zones. The obtained results are indicative of a decrease in the oscillations of velocity in the areas outside the surf zones. In this case, the results of the analytical model are also in good agreement with results of the numerical model as the value of $L_{\text{h}}$ reaches the lowest values in the offshore zone. Also, based on the equation 8, the undertow velocity reaches the minimum level in the offshore zones.

Dalrymple and Li used some experimental and numerical studies to investigate the instabilities of the undertow currents and reached some similar conclusions. They found that the vorticities have begun decaying in the offshore areas because of the profile of the undertow velocity had become linear at these points. Based on the experimental studies, they also found out that the steady undertow can be unstable to slow oscillations outside of the surf zone.

**Conclusion**

In this study, an analytical model can predict the magnitude and intensity of undertow velocity in a beach with different slopes and hydrodynamic conditions. Increase of the forcing intensity($\gamma$) increases the undertow velocity. But if $\gamma$ violates a specific limit, the undertow velocity reduces and this also changes of current oscillations. In addition, this study shows that the increase of beach slope causes the increase of undertow velocity and the increase of the number of current oscillation frequency in a specific period of time. So, the oscillations of the undertow velocity are proportional to the hydrodynamic conditions governing the beach.

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**References**

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