Transformation of a solitary wave propagating over a Shelf

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Received 4 February 2013; revised 25 February 2014

Transformation of single solitary wave passing over a semi-infinite shelf is investigated in this study. Considering the energy loss near the corner of the step, a semi-analytical solution to both the transmission and reflection coefficients is derived. Since the solitary wave will decompose to several solitons in the far field, the amplitudes of separated solitons are discussed as well. The results are very identical to those shown in previous studies. Furthermore, the appearance oscillatory tails appearing after split solitons are investigated by using the bilinear transformation method. Present calculations are very consistent with numerical and experimental results given by existing studies.

[Keywords: solitary wave, Transmission coefficient, Reflection coefficient, Split solitons, Oscillatory tails]

Introduction

The transformation of single solitary wave propagating over a semi-infinite shelf is an important topic in ocean engineering, fluid mechanics and many academic fields. The reason why people are interested in this may be the similarity between tsunami propagation and present problem. As well known when tsunamis propagate from deep water to a continental slope, the rapid change of bottom topography usually results in an essential transformation of incoming long waves. The transmission, reflection and fission of long waves thus attract a great deal of attentions from oceanographers.

In Lamb’s famous book in fluid mechanics, small-amplitude waves travelling over an abrupt step were studied. The results showed that both the transmission and reflection coefficients depend only on the depth ratio. For studying nonlinear effects inducing from the rapid change of topography, Peregrine derived the equations of motion for small-amplitude waves including nonlinear terms over a beach of uniform slope and conducted numerical simulations. It shows that the amplitude of incoming wave will gradually change and reflected waves will simultaneously occur as well. Later, Madsen and Mei investigated the case of nonlinear long waves propagating over a step through a mild slope. According to their analytical and numerical results, the front face of the solitary wave steepens when it encounters the slope. It is found that the transmitted wave disintegrates into several individual solitons on the shelf followed by a train of oscillatory tails. Tappert and Zabusky analyzed a nonlinear dispersive wave travelling onto a step through a slope. They assumed that the length scale of the slope region is much smaller than those on which nonlinearity and dispersion act. Their theoretical results may be applied to the case of an abrupt shelf. Johnson provided numerical model to calculate oscillatory tails behind split solitons that had not been thoroughly discussed previously. According to his investigation, oscillatory tails will vanish under some specific depth ratios. This phenomenon is named as the exact soliton mode. Moreover, the fission of solitary waves was also studied.

Mei derived the transmission and reflection coefficients by applying a semi-empirical theory with the energy loss due to the junction without the consideration of nonlinear effects near the step. Seabra-Santos et al. presented both numerical and experimental results indicating that the amplitude ratios of split solitons are independent of the normalized amplitude of the incident wave. This observation closely corresponds to previous theoretical studies except for Mei’s results. In a related work, Losada et al. distinguished four types of wave evolution in the transmitted region. The breaking state was observed as well. Goring and Raichlen provided an experiment to verify the nonlinear effect. It shows that nonlinear effects become important as the
propagation distance increases. Eventually in the far field, dispersive effects will become important as well. Briefly, aforementioned studies have provided important and abundant results for present problem. However, from the viewpoint of theoretical analysis, it still requires great efforts to establish a model to calculate the transmission and reflection coefficients and even oscillatory tails more accurately. Present work focuses mainly on establishing a semi-theoretical model to describe this problem. The construction of this paper is as below. In Section 2, the amplitudes of transmitted and reflected waves in the neighborhood of the junction are solved. Both the transmission and reflection coefficients are thus obtained. Split phenomenon of the transmitted wave on the shelf is discussed in Section 3. Amplitude of each split soliton and the number of waves are also calculated. In Section 4, the occurrence of oscillatory tails behind those split waves is solved by using the bilinear transformation. State of the exact soliton mode is also explained by comparing our model to previous numerical study. Conclusions are made in the last section.

Materials and Methods

Mathematical Formulation

In this section, a semi-theoretical solution for single solitary wave propagating over a semi-infinite step is demonstrated, as shown in Fig.1. As a solitary wave approaches an infinite step, strongly interacting effects will occur in the neighborhood of the junction. The magnitude of the neighborhood region may be about one or two wave lengths from the junction but excluding the immediate vicinity one or two depths from the junction. Firstly, the Korteweg-de Vries equation is introduced

\[
\eta_t + \sqrt{gh_0} \left(1 + \frac{3}{2} \frac{\eta}{h_0}\right) \eta_x + \frac{1}{6} \sqrt{gh_0} h_0^2 \eta_{xxx} = 0, \quad \ldots (1)
\]

where \( \eta \) denotes the elevation of free surface, \( h_0 \) the water depth and \( g \) the gravity. The solution of Eq.(1) for solitary waves going along the positive x-axis is

\[
\eta = \eta_0 \text{sech}^2 k(x - C_0 t), \quad \ldots (2)
\]

where \( C_0 \) is the wave speed. Based on KdV equation and the characteristics of solitary waves, the wave profiles on both sides of the abrupt junction located at \( x = 0 \) are assumed to be

\[
\eta_i = a_i \text{sech}^2 k_i(x - C_i t) \quad (x < 0), \quad \ldots (3)
\]

\[
\eta_r = a_r \text{sech}^2 k_r(x - C_r t) \quad (x > 0), \quad \ldots (4)
\]

where

\[
k_i = \sqrt{\frac{2a_i}{4h_i^3}}. \quad \ldots (5)
\]

\[
C_i = \sqrt{gh_i \left(1 + \frac{1}{2} \frac{a_i}{h_i} \right)}.
\]

\[
C_r = \sqrt{gh_r \left(1 + \frac{1}{2} \frac{a_r}{h_r} \right)}.
\]

The subscripts \( i, t \) and \( r \) denote the properties of the incident, transmitted and reflected waves, respectively. From Eq.(2) to Eq.(4), the following relation can be readily obtained

\[
k_i = \frac{C_i k_i}{C_t}. \quad \ldots (5)
\]

Now two conditions are required to solve the transmitted and reflected wave profiles. First, the continuity of volume flux must be consistent at \( x = 0 \) which results in

\[
\left( \frac{a_i - a_r}{C_i} \right) h_i = \frac{a_i}{C_i} h_2. \quad \ldots (6)
\]

Next, the transmission and reflection coefficients defined as

\[
T = \frac{a_t}{a_i}, \quad \ldots (7)
\]

\[
R = \frac{a_r}{a_i}, \quad \ldots (8)
\]

are substituted into Eq.(6). It reads

\[
\frac{2h_i a_i}{2h_i + a_i} - \frac{2h_i a_r}{2h_i + a_r} = \frac{2h_i a_t^2}{2h_i + a_i} \frac{h_2}{h_i}. \quad \ldots (9)
\]
While considering the strong hydraulic loss near the junction, the following semi-empirical formula must be adopted

$$\eta_1 - \eta_2 = C_i u_2$$  \hspace{1cm} \ldots (10)

where

$$C_i = \frac{2}{5} \frac{faT}{C_i}$$

and $f$ represents the friction coefficient affected by the geometric shape of the junction. Substituting Eqs.(3) and (4) into Eq.(10) leads to

$$1 + R = T \left(1 + \frac{2 faT}{h_2} \left(1 + \frac{aT}{2h_2} \right)^2 \right)$$  \hspace{1cm} \ldots (11)

Now the coefficients $T$ and $R$ can be readily obtained by solving Eqs.(9) and (11). Figure 2 shows the variation of the transmission coefficient with the depth ratio under various incident wave heights. Mei’s results revealed that the transmission coefficient depends on both depth ratios and the relative incident wave heights. In our study, the coefficient $T$ is almost independent of the relative incident wave height while the depth ratio is large enough. In the next section, we shall verify that our results are much closer to experimental investigations than Mei’s results. Figure 3 describes the relation between the reflection coefficient and the depth ratio. It is obvious that the value of $R$ increases as the depth ratio $h_2/h_1$ decreases.

**Results and Discussions**

**Fission of Transmitted Soliton**

In this section, the fission of transmitted wave in the far field (on the step far away from the corner) will be investigated. First, we introduce the following transformations

$$\zeta = -\frac{2}{\eta_0} \eta \quad X = \frac{3\eta_0}{4h_0^2} \left(x - \sqrt{gh_0} t \right)$$

$$\tau = \frac{\eta_0}{16h_0^2 \sqrt{3gh_0}} \cdot t$$  \hspace{1cm} \ldots (12)

Applying Eq.(12) into Eq.(1) yields the canonical KdV form

$$\zeta_x - 6 \zeta \zeta_x + \zeta_{xxx} = 0$$  \hspace{1cm} \ldots (13)

Based on the transformations in Eq.(12), Eq.(4) can be rewritten as

$$\xi_2 = -\left(2T \left(\frac{h_1}{h_2}\right)^2\right) \sech^2 X$$  \hspace{1cm} \ldots (14)

Equation (14) can be also expressed in another form

$$\xi_2 = -p(p+1) \sech^2 X$$  \hspace{1cm} \ldots (15)
In order to determine the wave heights and the number of split solitons, we have to solve the eigenvalues, $\lambda_n$, of the one-dimensional time-independent Schrödinger equation. According to Zabusky’s study, the total number of split solitons, $N$, must satisfy the following relation

$$N - 1 \leq p < N,$$  \hspace{1cm} \ldots (16)

where

$$p = \frac{1}{2} \left(1 + \sqrt{1 + 8 \left( \frac{h}{h_0} \right)^2} \right)$$

is obtained by balancing Eqs. (14) and (15). Besides, the corresponding amplitude ($a_j$) of the $j$-th split soliton is

$$a_j = \left( \frac{h_2}{h_1} \right)^2 \cdot (p + 1 - j)^2 \quad (j = 1, 2, \ldots, N).$$  \hspace{1cm} \ldots (17)

The variation of the amplitude of each split soliton with the depth ratio is plotted in Figure 4. It indicates that more solitons are generated as the depth ratio decreases. At least two split solitons occur on the step except in the case of $h_1 = h_2$. Table 1 shows the relation of the number of split solitons with the depth ratio according to the results of Fig. 4. Besides, as the transmission coefficient $T$ is nearly independent of the relative incident wave height for larger depth ratios due to Fig. 2, the number of split waves is independent of the incident wave height as well. Figure 5 describes some theoretical and experimental results concerning the number of split solitons for various depth ratios. Notice that the amplitudes of split solitons of all theoretical studies are independent of the incident wave height except Mei’s solution. While considering the separation loss near the junction, the wave amplitudes of split solitons predicted by our model are lower than those of other analytical investigations and are much closer to the experimental measurements. Figure 6 shows that only one reflected wave occurs.

**Oscillatory tails**

Though the number of split waves has been calculated in previous section, the cause of occurrence of oscillatory tails, which appear behind those transmitted solitons and are observed in both earlier numerical and experimental studies, has not been well investigated by theoretical methods. Herein we apply a bilinear transformation which usually concerns the problem of uni-directional solitons into KdV equation to account for why these tails exist. First, we begin with another canonical form of KdV equation

$$\eta_t + 6\eta \eta_x + \eta_{xxx} = 0.$$  \hspace{1cm} \ldots (18)

The logarithmic transformation for solving KdV equation is introduced

$$\eta = 2 \ln F_{xx}.$$  \hspace{1cm} \ldots (19)

Applying this transformation to Eq. (18) gives

$$F_{t} (F_{x} + F_{xxx}) - F_{xx} (F_{x} + F_{xxx}) + 3F_{x}^2 - F_{x} F_{xxx} = 0.$$  \hspace{1cm} \ldots (20)

Since Eq. (20) is hard to solve directly, a bilinear transformation is thus introduced

$$D_x^m D_{x'}^n f \cdot g = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n f(x, t) g(x', t') \bigg|_{x = x'} \bigg|_{t = t'},$$  \hspace{1cm} \ldots (21)

$\eta = 0$.

Table 1—The number of split solitons versus the depth ratio

<table>
<thead>
<tr>
<th>Range of Depth Ratio</th>
<th>Number of Solitons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>1</td>
</tr>
<tr>
<td>0.60-1.00</td>
<td>2</td>
</tr>
<tr>
<td>0.43-0.60</td>
<td>3</td>
</tr>
<tr>
<td>0.33-0.43</td>
<td>4</td>
</tr>
<tr>
<td>0.28-0.33</td>
<td>5</td>
</tr>
<tr>
<td>0.24-0.28</td>
<td>6</td>
</tr>
<tr>
<td>0.21-0.24</td>
<td>7</td>
</tr>
<tr>
<td>0.18-0.21</td>
<td>8</td>
</tr>
</tbody>
</table>

Fig. 4—The number and amplitudes of split transmitted solitons.
where $m$ and $n$ are integers and $D$ denotes the differential operator. Applying the bilinear transformation into Eq.(20), it results in

$$D_x (D_1 + D_x^3) F \cdot F = 0.$$  

(22)

In order to simplify Eq.(22), the operator, $\Pi$, is defined as

$$\Pi (D_1, D_2) = D_2 (D_1 + D_x^3),$$  

(23)

where $D_i$ can represent either the constant or the differential operator. To solve such a complicated and highly nonlinear equation, the unknown $F$ is assumed as

$$F = 1 + \sum_{n=1}^{\infty} f_n,$$  

(24)

where

$$f_i = \sum_{j} \exp \left( a_i x - a_i^3 t \right),$$  

(25)

indicates characteristics of each soliton in a summation.
form. The wave height of each split wave is $a_j^2 / 2$ and the integer $N$ represents the number of split solitons. Now we apply Eqs.(23) and (24) into Eq.(20), it yields

$$
\prod (D_{1},D_{x}) (f_1 \cdot 1 + f_1) + \prod (D_{1},D_{x}) (f_2 \cdot 1 + f_1 \cdot 1 + f_2)
$$

$$
+ \ldots + \prod (D_{1},D_{x}) \left( \sum_{i=0}^{N} f_i f_{n,j} \right) + \ldots = 0,
$$

(26)

Obviously, one can use the iterative skills to solve $f_i$ in Eq.(26) by setting each term to be zero. It is found that only those $f_j$ terms ($j \leq N$) will be nonzero if there exist $N$ split solitons.

Before the cause of occurrence of oscillatory tails is discussed, some important properties of solitons propagating in the same direction should be reviewed. First, since the basic characteristic of a solitary wave is that the wave speed is proportional to its amplitude, the “larger” soliton will overtake the “smaller” one as the collision happens. Next, when time $t$ approaches infinity, these split solitons will be well developed with their individual wave profiles and line up in sequence. Furthermore, the strongest interaction occurs at $t = 0$ where all solitons are segregated from a single solitary wave. Hence one can integrate those solitons into a single one with the wave height $a_T$ exactly at the junction of the step. That is to say that this single soliton will disintegrate into $N$ individual solitons which gradually regain their original wave profiles as time $t$ goes by. Now the amplitude of each split wave and the difference between $a_T$ and $a_i$ are shown in Fig. 7. The gap between $a_T$ and $a_i$ is the “source” of oscillatory tails. It is also observed that the gap will vanish when the depth ratio is 0.596 and 0.423. The state where all oscillations vanished was named as “the exact soliton mode” $^5$. The comparison of the state of the exact soliton mode given by present study and Johnson’s solution is listed in Table 2. The results of analytical and numerical studies are very coincident and the difference is less than 6%. In addition, from Fig. 7, the amplitude of the strongest soliton, $a_{i1}$, is always larger than $a_i$ and $a_T$. This state also demonstrates the run-up phenomenon will occur in the far field.

Conclusions
In present study, an semi-theoretical model for calculating the wave evolution when single solitary wave propagates over a semi-infinite step is provided. While considering the separation loss in the neighborhood region of the junction, the transmission and reflection coefficients are derived. The transmitted coefficient is approximately independent of the relative incident wave height for larger depth ratio.

In the far field, the transmitted soliton will gradually split into several solitons. The number of scattering solitons depends mainly on the depth ratio. Behind those split solitons, oscillatory tails arise as well. As the bilinear transformation is applied, our results show that the source of these oscillations may be attributed to the difference between $a_i$ (the transmitted wave amplitude) and $a_T$ (the accumulated amplitude). It is found that those tails
will vanish at specific depth ratios which are called the exact soliton modes. The disappearance of oscillations predicted by our model and Johnson’s numerical solution is very consistent.

Acknowledgement

Author is indebted to National Science Council of Taiwan for financial support with I-006-302 grant NSC 101-2911-2221-E-270-001-MY2.

References