New Method to Study the Area of Fibres in Yarn Cross-section

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A new method of determining the area of fibres in the yarn cross-section is proposed. In this, the centre of gravity of the yarn cross-section is calculated on the basis of the \(x,y\) coordinates of fibres in yarn cross-section, and then the area of cross-section is divided into class-intervals of equal width. A mathematical model to study the radial packing density of yarn is also proposed. To calculate the area of fibres in the cross-section a program was made on HP 9810 A, which is based on the theory described in this paper. The method is also suitable for studying the tendency of inner and outer migrations of different types of fibres in blended yarn.

Keywords: Fibre cross-sectional area, Fibre migration, Radial packing density, Yarn cross-section

1 Introduction

Yarn structure has been studied by many researchers and much work has been done on the relation between yarn structure and properties. Many of the workers have used the tracer fibre technique of Morton\(^1\) to study the longitudinal fibre properties for arriving at yarn structure. However, not much information is available in the literature on the cross-sectional cutting method of studying the yarn structure. Although tedious and time-consuming, this method gives valuable information on yarn structure. In this paper, we propose a new method, SECANT (Systematic Experimental and Computing Analysis of Textiles) to study the yarn cross-section. In this, the centre of gravity of yarn cross-section is first calculated and then the area of cross-section is divided into class-intervals of equal width. According to Hearle et al.\(^2\), however, it is preferable to use zones of equal area so that the fibres are equally distributed between all zones. On the other hand, according to Zotikov and Trolova\(^3\), the results obtained by using both the methods should be identical for the same cross-section. These authors based their calculations on the fibre-number density per unit area, where variation in fibre cross-sectional area is negligible.

For fibres like wool and cotton of any given diameter, the variation in fibre cross-sectional area is quite large, and a method based on fibre density per unit cross-sectional area of the yarn is therefore not applicable. In the SECANT method the radial packing density of yarn is measured as the ratio of the cross-sectional area of fibre in a given zone of the yarn cross-section to the area of that zone. Hence the method is applicable to all types of fibrous materials. The theoretical part of this method, which is based on the doctoral work of one of us (SMI)\(^4\), is described in this paper.

In the SECANT method, the yarn cross-section is divided into 20 zones of equal width. Earlier workers had considered only five zones in their calculation. Division of yarn cross-section into 20 zones gives a better picture of radial packing density. The advantages of dividing yarn cross-section into equal width zones are that the cross-section of yarn can be divided into a number of zones and that it also gives the exact value of radial packing density. In the constant-area method, the first three zones of bigger size do not give an exact idea of packing density near the yarn axis, which is very important; besides, the last zones are very small. Hence, the constant zone width method was used to study the packing density of yarn. To calculate the area of fibres in the class intervals, a program was made. The theory is described in the following text.

2 Analysis of Yarn Cross-section with the Conception of SECANT Method

Geometrically, it is possible to arrive at some relationship between the distances of fibres, area and angle of inclination of fibres in the single yarn cross-section. For making a preliminary analysis of cross-section it is necessary to get the position and coordinates of each and every fibre in the yarn cross-section. It is difficult to locate the axis of yarn because fibres are not uniformly distributed around the yarn axis. In the SECANT method the centre of gravity of yarn cross-section is determined by the position of

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fibres in yarn cross-section and a set of these values was used to locate the yarn axis.

Let us consider a general cross-section of yarn, consisting of a number of fibres with centre coordinates \((x_i,y_i)\) \((i = 1, 2, 3 \ldots j \ldots n)\). The quadratic distance between \(i\)-th and \(j\)-th centres of fibres is:

\[
(x_i - x_j)^2 + (y_i - y_j)^2
\]

The average quadratic distance of all fibres in the yarn cross-section from the centre of \(j\)-th fibre is given as follows:

\[
\bar{p}_j = \frac{\sum_{i=1}^{n} [(x_i - x_j)^2 + (y_i - y_j)^2]}{n - 1}
\]

where

\[
\bar{p}_j = \frac{\left\{ (\bar{x}^2 + \bar{y}^2 - 2x\bar{x} + 2y\bar{y}) \right\}}{n - 1}
\]

\[
\bar{x}^2 = \frac{\sum_{i=1}^{n} x_i^2}{n}, \quad \bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}
\]

The value \(\bar{p}_j\) represents the critical distance from the centre of \(j\)-th fibre to the centre of other fibres. On the basis of the above hypothesis, 10% of total fibres in yarn cross-section on the outer surface of yarn due to hairiness gives practically a negligible error to calculate the centre of gravity of yarn cross-section. Suppose the coordinates of yarn axis are \((x_0, y_0)\), then the distance from the centre of \(i\)-th fibre to the yarn axis is

\[
r_i = \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2}
\]

The inclination of \(i\)-th fibre situated at \(r_i\) distance to the yarn axis is

\[
\phi_i = \cos^{-1}\left[ \frac{(x_i - x_0)/r_i}{(y_i - y_0)/r_i} \right]
\]

The set of \((r_i, \phi_i)\) represents the polar coordinates of the centre of fibre from the point O on yarn axis.

3 Distribution of Area of Fibres into Class-intervals

In reality the cross-section of a particular fibrous material has its own shape. But in this work it is assumed that fibres have a circular cross-section with diameter \(d\). In general the diameter of fibre can be calculated by the following relationship:

\[
s = t/\rho = \frac{\pi d^2}{4}
\]

\[
d = \sqrt{4t/\pi \rho}
\]

where \(t\) is the fineness of fibre in tex, and \(\rho\), the specific density of fibre in kg m\(^{-3}\).

Consider a typical class-interval with respect to yarn axis with lower limit \(r_a\) and upper limit \(r_b\) in which the cross-section of the ideal fibre is placed, as shown in Fig. 1, where point \(s\) represents the centre of \(i\)-th fibre placed at a distance \(r_i\) from the yarn axis.

In the given case \(r_a \geq (r_i - d/2)\)
\(r_b \leq (r_i + d/2)\)
\(r_a < r_b\).

If \(P\) is the area of fibre in the \(j\)-th class-interval, then

\[
P = P^* - P_1 + P_2
\]

where \(P^*\) is the area of circular part lying between \(U_sV_a\) and \(U_sV_b\) straight lines; \(P_1\), the area of section, which is cut by the straight line \(U_sV_a\) and the arch of radius \(r_a\) and \(P_2\), the area of section, which is cut by the straight line \(U_bV_s\) and the arch of radius \(r_b\).

3.1 Calculation of Values \(r_a^*\) and \(r_b^*\)

For the values of \(r_a^*\) and \(r_b^*\) cosine relation can be used. For the value \(r_a^*:\)

\[
r_a^* = r_a - 2r_x^* \cos \psi_a = (d/2)^2
\]

On rearranging

\[
\cos \psi_a = \frac{r_a^2 + r_b^2 - (d/2)^2}{2r_x^*}
\]

For the value \(r_b^*:\)

\[
r_b^2 = r_b^2 + 2r_x^* \cos \psi_b = (d/2)^2
\]

On rearranging

\[
\cos \psi_b = \frac{r_a^2 + r_b^2 - (d/2)^2}{2r_x^*}
\]

From this it follows that

\[
r_a^* = r_a \cos \psi_a
\]

\[
r_b^* = r_b \cos \psi_b
\]

3.2 Calculation of Area \(P^*\)

Although the area \(P^*\) can be calculated directly with the help of geometrical relations, in this work the area \(P^*\) is derived integrally. The equation for circular fibre is given as follows:

\[
(r - r)^2 + y^2 = (d/2)^2
\]

then

\[
y = \sqrt{(d/2)^2 - (r - r)^2}
\]
For area $P^*$, it follows

$$P^* = 2 \int_{r_2}^{r_1} y \, dr$$  \hspace{1cm} \text{(13)}

On putting the value of Eq. (12) in Eq. (13), we get

$$P^* = 2 \int_{r_2}^{r_1} \sqrt{(d/2)^2 - (r-r^*_1)^2} \, dr$$

At values

$$t = r - r_1$$

$$dt = dr$$

On rearranging

$$P^* = 2 \int_{r_2}^{r_1} (r_2 - r) \sqrt{(d/2)^2 - (r-r^*_1)^2} \, dr$$

$$= \left\{ \frac{1}{2} t \sqrt{(d/2)^2 - t^2} + (d/2)^2 \sin^{-1} t/d/2 \right\} (r_2^*_1 - r_1)$$

$$P^* = (r_2^*_1 - r_1) \sqrt{(d/2)^2 - (r_2^*_1 - r_1)^2}$$

$$+ (d/2)^2 \sin^{-1} \left( \frac{r_2^*_1 - r_1}{d/2} \right)$$

$$+ \sqrt{(d/2)^2 - (r_2^*_1 - r_1)^2} - (d/2)^2 \sin^{-1} \frac{r_2^*_1 - r_1}{d/2}$$  \hspace{1cm} \text{(14)}

3.3 Calculation of Area $P_1$ and $P_2$

For calculating the area $P_1$ the following relation is used

$$r^*_a = r_a \cos \psi_a$$

$$P_1 = \pi r_a^2 \psi_a - r_a^* (r_a \sin \psi_a)$$

On rearranging

$$P_1 = r_a^2 (\psi_a - \sin \psi_a \cos \psi_a)$$  \hspace{1cm} \text{(15)}

Similarly, area $P_2$ can be calculated as:

$$P_2 = r_a^2 (\psi_a - \sin \psi_a \cos \psi_a)$$  \hspace{1cm} \text{(16)}

3.4 Calculation of Area $P$

From Eqs (4), (14), (15), (16), (9) and (10), we get

$$P = B - A$$

where

$$B = (r_b \cos \psi_b - r_i) \sqrt{(d/2)^2 - (r_b \cos \psi_b - r_i)^2}$$

$$+ (d/2)^2 \sin^{-1} \left( \frac{r_b \cos \psi_b - r_i}{d/2} \right)$$

$$+ r_i^2 (\psi_b - \sin \psi_b \cos \psi_b)$$

$$A = (r_a \cos \psi_a - r_i) \sqrt{(d/2)^2 - (r_a \cos \psi_a - r_i)^2}$$

$$+ (d/2)^2 \sin^{-1} \left( \frac{r_a \cos \psi_a - r_i}{d/2} \right)$$

$$+ r_i^2 (\psi_a - \sin \psi_a \cos \psi_a)$$

Thus, $P$ gives the area of a circular fibre and this value agrees with geometrical derivations also.

3.5 Some Special Cases

First case:

Let $r_b = r_1 + d/2$

According to Eq. (8), it follows

$$\cos \psi_b = \frac{r_1^2 + r_i^2 + r_i d + (d/2)^2 - (d/2)^2}{2r_1^2 - r_i d} = 1$$

which means

$$\psi_b = 0$$

further

$$B = (d/2) \sqrt{(d/2)^2 - (d/2)^2} + (d/2)^2 \sin^{-1} 1 + 0$$

$$= (d/2)^2 \pi/2 = \frac{\pi d^2}{8}.$$  \hspace{1cm} \text{(17)}

Second case:

Let $r_a = r_1 - d/2$.

From Eq. (6), it follows

$$\cos \psi_a = \frac{r_1^2 + r_i^2 - r_i d + (d/2)^2 - (d/2)^2}{2r_1^2 - r_i d} = 1$$

which means $\psi_a = 0$

further

$$A = -(d/2) \sqrt{(d/2)^2 - (- d/2)^2}$$

$$+ (d/2)^2 \sin^{-1} (-1) + 0 = -(d/2)^2 \frac{\pi}{2}$$

$$= -\frac{\pi d^2}{8}.$$  \hspace{1cm} \text{(18)}

Third case:

Let $r_b = r_1 - d/2$.

which means the fibre lies only in one class-interval.

Then according to the cases 1 and 2, we get

$$B = \frac{\pi d^2}{8}$$

$$A = -\frac{\pi d^2}{8}$$

as

$$P = B - A$$

$$P = \frac{\pi d^2}{4}.$$  \hspace{1cm} \text{(19)}

These relations allow us to distribute the area of circular fibre into individual class-intervals. Consider, for example, that the class-intervals are of the same constant width $h$ in increasing order of radius. If class-intervals are indicated by $j$ ($j = 1, 2, 3, ...$), then the upper radius of $j$-th interval is equal to $jh$, lower radius is $(j - 1)h$. If the centre of $i$-th fibre is at radius $r_i$ from yarn axis as shown in Fig. 1 then radius $(r_i - d/2)$ lies in $j_1$ (lower class interval) and radius $(r_i + d/2)$ lies in $j_a$ (upper class interval). Class-intervals between these
class-intervals are indicated by intermediate class-intervals.

If $j_1 = j_n$, it means the whole fibre lies in one class-interval and its total area $\pi d^2/4$ will belong to this particular class-interval. If $j_1 = j_{n+1}$, then the intermediate class-interval does not exist and the area of fibre is distributed only in the two class-intervals. It is possible to determine the part of circular area belonging to lower class-interval, for this upper limit of lower class-interval is indicated by the value $r_n$, and radius $r_1 - d/2$ for the value $r_b$ and earlier cited special case is used to calculate the value of $P$. Similarly, it is possible to calculate the area of circle belonging to upper class-interval by considering $r_a$ is lower limit of upper class and $r_b = r_1 + d/2$ (this can be explained by the first special case). In practice it is sufficient to calculate only the area belonging to one of these class-intervals.

In general cases, one or more class-intervals exist. The area belonging to intermediate class-interval can be calculated by the general case, where $r_a$ is lower and $r_b$ is upper radius of given class-interval. The area of fibre in lower and upper class-intervals is calculated as described above.

The following conditions must be satisfied for all the cross-sections of yarn.

The area of cross-section is divided into 20 class intervals with increasing radius of same width. The last interval, i.e. 20th, consists of the area of fibre belonging to this class-interval and also the area of fibre which falls outside the 20th class-interval.

The width of class-intervals should not be less than the radius of fibre.

4 Correction of Fibre Cross-sectional Area

The cross-sectional area of each fibre lies around its centre. Its value and shape depend on profile and inclination of fibre. In this method, the area of fibre is calculated by the following conception.

Consider an ideal cylindrical fibre, parallel to yarn axis and its lateral cross-sectional area similar to circle. If its centre lies at radius from yarn axis, then cross-sectional area of this fibre will not be a circle, but it will be an ellipse and the area of this ellipse will be greater than the area of circle. In reality, fibres are not parallel to yarn axis. Inclination of fibres is due to (i) yarn twist, and (ii) migrational behaviour of fibre.

Consider the helical model of yarn. If $\beta_1$ is the angle of fibre on yarn axis and its centre is situated at distance $r_1$ from yarn axis as shown in Fig. 2, the angle $\beta_1$ can be described by the well-known equation:

$$\tan \beta_1 = 2\pi r_1 Z$$

where $Z$ is the fibre twist in yarn.

The minor axis of the ellipse will remain the same as fibre diameter and the major axis of elliptical cross-section is given by the following relation:

$$a_i = \frac{d}{\cos \beta_1}$$

After substitution of value $\beta_1$, we get

$$a_i = d \cdot \sqrt{1 + (2\pi Z)^2}$$

The area of this ellipse is $\pi a_i d / 4$ and is greater than area of a circle $\pi d^2 / 4$. The ratio of area of ellipse to the area of circle is given as follows:

$$\omega_i = \frac{\pi a_i d}{4} / \frac{\pi d^2}{4} = \frac{a_i}{d} \cdot \frac{1}{\cos \beta_1}$$

$$\omega_i = \sqrt{1 + (2\pi Z)^2}$$

During the calculation of the area of fibre, the first cross-section of fibre was considered as circle and this area was distributed into class-intervals and after that this area was multiplied by the corresponding value of $\omega_i$. By considering this correction, the actual area of fibre can be calculated in class-intervals.

5 Conclusions

(1) The SECANT method is applicable to all types of fibrous materials as well as to blended and plied yarns.

(2) The cross-section of yarn can be magnified up to 500x, which is helpful to observe individual fibres in yarn cross-section.

(3) The area of fibres in cross-section is calculated after punching the $x$ and $y$ coordinates of fibres with the help of a Digimet, which is important to calculate the centre of gravity of yarn cross-section and the area of fibres.

(4) The method is suitable for studying the tendency of inner and outer migrations of different types of fibres in blended yarn.

(5) The cross-section of yarn is divided into class-intervals with constant width of class-intervals, which gives the exact conception of radial behaviour of packing density.

(6) Punched tape can be fed to the table calculator
HP 9810 A where the program is used on the basis of theoretical relations and where all values are obtained in the form of tables and graphs.

References
1 Morton W E, Text Res J, 26 (1956) 325.