Iterative algorithm for microwave tomography using Levenberg-Marquardt regularization technique

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An inverse iterative algorithm for microwave imaging based on moment method solution is presented here. The iterative scheme has been developed on constrained optimization technique and is certain to converge. The algorithm is applied to synthetic data and the results of reconstruction of complex permittivity distribution show a very high degree of accuracy. Levenberg - Marquardt method solves the ill posedness of the problem.

Keywords: Iterative algorithm, Microwave tomography, Levenberg-Marquardt method

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1 Introduction

Microwave tomography techniques for biomedical applications are lagging behind imaging schemes based on X-ray, ultrasound, nuclear magnetic resonance (NMR) and even electrical impedance tomography (EIT). During the past 20 years, immense research is being carried out in microwave tomography to quantitatively reconstruct the complex permittivity distribution of the biological media. Spectral methods used for diffraction tomography have been investigated with application to microwave. The advantage of such methods results from the existing fast numerical algorithm. However, the diffraction tomography suffers from the fact that it is marred by strongly inhomogeneous media where Born and Rytov approximations are not valid. The other methods based on moment method solutions are being explored rigorously, but the stability depends on the measurement accuracy due to ill conditioning of the matrix.

In our earlier works, we had suggested quasi-ray optic SIRT-style algorithms for microwave imaging. In those first generation algorithms, it was assumed that only those cells situated within the beamwidth of the transmitter radiation pattern, would effectively contribute to the field at the end of a ray. Those linear and non-linear algorithms did not reconstruct the image quantitatively to the extent, which could be considered to be clinically important. In this paper, an iterative algorithm based on moment method solution has been suggested taking into account the contributions of all the cells within the model. The aim of this algorithm is to minimize the difference between the measured scattered field and the scattered field calculated from the numerical model.

2 Forward Problem

Let us consider a cylindrical object of arbitrary cross-section characterized by a complex permittivity distribution \( \varepsilon(x,y) \). It is assumed that the constitutive parameters of the scatterer do not vary along the axis of the cylinder. The cylinder is kept in free space. The object is illuminated by an electromagnetic wave radiated from an open-ended waveguide for which the incident electric field \( E^{inc} \) is parallel to the axis of the cylinder.

Let \( E \) represents the total electric field and \( E^s \) represents the scattered field, which is generated by the equivalent electric current radiating in free space so that:

\[
E = E^{inc} + E^s
\]  \ ...(1)

where the equivalent electric currents are defined by:
with
\[ k_x = k \sqrt{\varepsilon} \quad \text{and} \quad k_y = \varepsilon_{0} \varepsilon_{0} \quad \text{(3)} \]
The total electric field can be calculated with an integral representation
\[ \vec{E}(x,y) = \vec{E}_{\text{inc}}(x,y) + \int \int \vec{J}_s(x,y) G(x,y;x',y') \, dx' \, dy' \quad \text{(4)} \]
where the Green’s function can be given by:
\[ G(x,y;x',y') = \frac{J}{4} H_0^2 \left( k \sqrt{(x-x')^2 + (y-y')^2} \right) \quad \text{(5)} \]
where \((x,y)\) and \((x',y')\) are the observation and source points respectively.

The solution of the forward problem (the computation of the total and scattered fields from the knowledge of geometrical and electrical properties of the object for a given incident field) is carried out by moment method\(^{15}\) using pulse-basis function and point matching technique.

### 3 Inverse Problem

The aim of the inverse problem is to find a stable solution \( \varepsilon^* \), which minimizes the squared error at the receivers:
\[ \phi(\varepsilon) = \| E(\varepsilon) - e \|_2^2 \quad \text{(6)} \]
where \(\dagger\) denotes the conjugate transpose and \(e \in \mathbb{C}^n\), the \(n\) electric fields are measured at receiver points, \(E: \mathbb{C}^m \rightarrow \mathbb{C}^n\), a function mapping the complex permittivity distribution with \(m\) degrees of freedom into a set of \(n\) approximate electric field observations, and also \(e \in \mathbb{C}^n\), the complex permittivity distributions with \(m\) degrees of freedom.

From Eq. (6), the gradient \(\nabla \phi = \phi'(\varepsilon)\) and the Hessian \(\mathbf{H} \phi = \phi''(\varepsilon)\) of the squared error at \(\varepsilon\) are
\[ \nabla \phi = E'(\varepsilon) \dagger (E(\varepsilon) - e) \quad \text{(7)} \]
\[ \mathbf{H} \phi = E'(\varepsilon) \dagger E'(\varepsilon) + \sum_i E_i''(\varepsilon) (E_i(\varepsilon) - e) \quad \text{(8)} \]
The second derivative term of Eq. (8) becomes negligible in the vicinity of minimum. Therefore
\[ \mathbf{H} \phi = E'(\varepsilon) \dagger E'(\varepsilon) \quad \text{(9)} \]
The Taylor series of \(\phi'(\varepsilon)\), taking into account only the linear terms, can be written as:
\[ \phi'(\varepsilon + \Delta \varepsilon) = \phi'(\varepsilon) + \Delta \varepsilon \phi''(\varepsilon) \quad \text{(10)} \]
In an attempt to minimize \(\phi\), an iterative procedure is followed for which a step \(\Delta \varepsilon\) is required such that \(\phi'(\varepsilon + \Delta \varepsilon) = 0\). Therefore,
\[ E'(\varepsilon) \dagger (E(\varepsilon) - e) + (E'(\varepsilon) \dagger E'(\varepsilon)) \Delta \varepsilon = 0 \quad \text{(11)} \]
from where the step \(\Delta \varepsilon\) can be written as:
\[ \Delta \varepsilon = (E'(\varepsilon) \dagger E'(\varepsilon))^{-1} E'(\varepsilon) \dagger (E(\varepsilon) - e) \quad \text{(12)} \]
During the \((i+1)^{\text{th}}\) iteration, the update values of the permittivities are
\[ \varepsilon_{i+1} = \varepsilon_i + \Delta \varepsilon_i \quad \text{(13)} \]

### 3.1 Computation of the Jacobian

The total fields \((E_i)\) at the center of the discretized cells are related to the incident electric field \((E_{\text{inc}})\) by the equation\(^{15}\):
\[ E_i = C^{-1} E_{\text{inc}} \quad \text{(14)} \]
where \(C\) is an \(n \times n\) coefficient matrix, \(E_i\) an \(n \times 1\) total field vector and \(E_{\text{inc}}\) is an \(n \times 1\) incident field vector in vacuum. The elements \(C_{m,n}\)'s are given by:
\[ C_{m,n} = \{ \pi k a_n H_1^{(2)}(ka_n) - 2j \} \quad \text{if} \quad p = m \quad \text{(15)} \]
\[ C_{m,n} = \{ j \pi k a_n / 2 \} (\varepsilon - 1) - 1 \} J_1(k a_n H_0^{(2)}(k a_n)) \quad \text{if} \quad p \neq m \quad \text{(16)} \]
Hence,
\[ \left[ E'_i \right] = \left( C^{-1} E_{\text{inc}} \right) / \partial \varepsilon_i \]
\[ = -C^{-1} \left( \partial C / \partial \varepsilon_i \right) C^{-1} [E_{\text{inc}}] \]
\[ = -C^{-1} \left( \partial C / \partial \varepsilon_i \right) [E_i] \quad \text{(17)} \]
To compute the Jacobian matrix \([E]'\), we notice that since the receivers are located at the center of some cells, hence \(E\) must be a subset of \(E'\). So, we can conclude \(E'\) is also a subset of \(E'_r\). In fact, we have first determined \(E'_r\), an \(n\times n\) matrix, for each transmitter position. We retained only \(q\) number of rows out of \(n\) rows, where \(q\) is the number of receivers for each transmitter position so that the resultant matrix would be of order \((q\times n)\). We appended \(q\) rows of each transmitter position so that the Jacobian matrix would be of order \((qs\times n)\), where \(s\) is the number of transmitter position. In our calculation, we have \(q = 18\), \(s = 16\) and \(n = 256\) so that the order of the Jacobian matrix was \((288\times 256)\).

4 Numerical Model

The theoretical model used to test our algorithm is shown in Fig. 1. It is a high contrast square biological object \(9.6 \text{ cm} \times 9.6 \text{ cm}\) consisting of muscle and bone having complex dielectric constants \(50-j23\) and \(8-j1.2\) respectively at a frequency of 1 GHz. The object is kept immersed in saline water having complex dielectric constant \(76-j40\). The target is illuminated with TE fields radiating from an open-ended dielectric filled waveguide having sinusoidal aperture field distribution. The transmitter is moved along four mutually orthogonal directions. For each of the transmitter positions along a particular transmitting plane, the received fields at eighteen locations in the other three orthogonal planes were measured theoretically at a frequency 1 GHz. Therefore, the measurement set contains 288 independent data. The rectangular model together with saline water region is divided into 324 equal square cells \(0.6 \text{ cm} \times 0.6 \text{ cm}\). For this selection, the cell dimension is roughly equal to \(0.15 \lambda_m\) in muscle and \(0.06 \lambda_b\) in bone at an assumed free space wavelength \(\lambda_0 = 30 \text{ cm}\). The dimensions of the bone regions are equal and given by \(1.8 \text{ cm} \times 1.2 \text{ cm}\). During the iterative reconstruction, the complex permittivity values of the cells filled up with saline water were assumed to be known, thus rendering the problem of estimating the complex dielectric constants of the remaining 256 cells. Since the number of independent equations (288) is greater than the number of unknowns(256), this is an over-determined case.

5 Results and Discussion

To apply the reconstruction algorithm, the biological medium was assumed to be a homogeneous one having complex dielectric constant \(50 - j23\) i.e. it was assumed to be filled up with muscle. The received fields at different receiver locations were computed for each transmitter position. The Jacobian matrix was computed. In Eq. (13), it was found that \((E'(\varepsilon)^\top E'(\varepsilon))^{-1}\) was ill conditioned, the condition number was found to be as large as \(10^{12}\). We choose Levenberg-Marquardt method to handle this ill conditioning.

The Marquardt form is given by:

\[
(A + \lambda I)\Delta \varepsilon = B \tag{18}
\]

where \(\lambda\) is a scalar, \(A = (E'(\varepsilon)^\top E'(\varepsilon))\) and \(B = E'(\varepsilon)^\top(E(\varepsilon)-\varepsilon)\).

The Marquardt method is as follows:

(i) \(\phi\) is computed; (ii) A value of \(\lambda\) is picked up \((\lambda = 0.0001)\) in our case; (iii) Eq. (13) is solved for \(\Delta \varepsilon\) and \(\phi^{k+1}\) is evaluated; (iv) If \(\phi^{k+1} \geq \phi^k\), \(\lambda\) is increased by a factor 10 and control is transferred to step (iii); (v) If \(\phi^{k+1} \leq \phi^k\), \(\lambda\) is decreased by a factor 10, the total solution is updated \(\varepsilon \leftarrow \varepsilon + \Delta \varepsilon\), and control is transferred to step (iii).

The only priori information we have used in our algorithm is that the real part of the complex dielectric constant cannot be negative and the imaginary part cannot be positive. Fig. 2 shows the model in terms of the real and imaginary parts. The plot of squared error
versus number of iterations is shown in Fig. 3. Figure 4 shows the reconstructed images after 0, 2, 7, 12, 14 and 24 iterations. We define the Mean Estimation Error as:

\[
\text{Mean Estimation Error} = \frac{1}{N} \sum_i \left| \frac{\varepsilon_i - \varepsilon_i^*}{\varepsilon_i^*} \right| \times 100 \% \quad (19)
\]

where \( N \) is the number of cells, \( \varepsilon_i \) is the estimated permittivity of the \( i^{th} \) cell and \( \varepsilon_i^* \) is the true permittivity of the \( i^{th} \) cell. The mean estimation errors after 25 iterations were as low as 10\% for both the real and imaginary parts.

The result shows that this iterative algorithm with Levenberg-Marquardt regularization provides us a far better reconstructed image than what we have obtained from our SIRT-style algorithm.
Fig. 4—Reconstructed images in terms of real and imaginary parts of the complex permittivity distributions: (a) & (g) after 0th iteration (i.e. initial guess), (b) & (h) after 2nd iteration, (c) & (i) after 7th iteration, (d) & (j) after 12th iteration, (e) & (k) after 14th iteration and (f) & (l) after 24th iteration respectively.

References