An approximate periodic solution for
\[ \frac{d^2y}{dt^2} + y = a + \epsilon y^2 \]

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The branch of mechanics that has been subjected to the most intensive attack from the nonlinear viewpoint is the theory of vibration of mechanical and electrical systems. Other branches of mechanics such as incompressible and compressible fluid flow, elasticity, plasticity, wave propagation, etc., also have been studied as nonlinear problems, but the greatest progress has been made in treating vibration of nonlinear systems.

The nonlinear differential equation
\[ y + y = a + \epsilon y^2 \]  (1)
subject to the initial conditions
\[ y = A, \dot{y} = 0 \text{ at } t = 0. \]  (2)
describe certain phenomena in the theories of general relativity, nonlinear vibrations, solid-state physics, the evolution of the radial motion of a satellite in the equatorial plane of an oblate planet, etc.\(^1\) Here \( \epsilon \) is a small parameter; \( a \) is a constant; and over dots denote differentiation with respect to the time, \( t \).

It is possible to obtain exact solutions for only a relatively few second-order nonlinear differential equations. They are exact in the sense that the solution is given either in closed-form or in an expression that can be evaluated to numerically to any desired degree of accuracy. Any second-order ordinary differential equation can be integrated numerically in a stepwise manner to yield a time-history of the motion. Numerical methods apply only to specific equations, and are not useful for general studies of behaviour of nonlinear vibrating systems. A large number of approximate analytical methods of nonlinear analysis exist (viz. averaging techniques, perturbation techniques, multiple-time scaling and harmonic balancing etc\(^3\)), each of which may or may not possesses advantages for certain class of problems. Some of these are restricted techniques which may work well with some types of equations, but not with others. In the perturbation method, the desired quantities are developed in powers of some parameter which is considered small, then the coefficients of the resulting power series are determined in a stepwise manner. The method is straightforward, although it becomes cumbersome for actual computations if many terms in the perturbation series are required to achieve a desired degree of accuracy. If the coefficients of nonlinear terms in the differential equation do not involve small parameters this method becomes inappropriate\(^4\). In such a situation, the harmonic balance method is more suitable for determining approximate periodic solutions.\(^5\) But, Nayefh and Mook\(^3\) states that, to obtain a consistent solution by using the method of harmonic balance, one needs either to know a great deal about the solution \( a \text{ priori} \) or to carry enough terms in the solution and check the order of the coefficients of all the neglected harmonics, and also cautions not to use this technique. These conclusions are drawn based on the assumption that the perturbation method provides the exact periodic solution. Infact, the potentiality of any approximate solution should be judged with exact solution of the problem.

The purpose of this study is to provide the periodic solution of the problem assuming two-term solution in the method of harmonic balance and confirm the results through exact integration for the applicable range of the amplitude, \( A \).

Since, the restoring force function, \( f(y) = y - a - \epsilon y^2 \), in the equation of motion (Eq. (1)) is a quadratic polynomial and hence a non-odd function, the behaviour of the oscillations is different for positive and negative amplitudes. The singular points of the differential equation (Eq. (1)) in \( y \) \( y \) plane obtained from the roots of \( f(y) \), are \((y_1, 0)\) and \((y_2, 0)\). Here \( y_{1,2} = (1 \mp \sqrt{1 - 4\epsilon a}) / 2\epsilon \) which are real only when \((1 - 4\epsilon a) > 0\). Since \( f'(y_1) > 0 \), and \( f'(y_2) < 0 \), the singular point \((y_1, 0)\) becomes a centre whereas the other point becomes a saddle point. Here, \( y \) prime denotes the differentiation with respect to \( y \). Therefore, the largest enclosed curve (viz. separatrix) in the phase-plane which passes just inside \((y_2, 0)\) is
\[ \frac{1}{2} y^2 + I(y) - I(y_2) = 0, \]  (3)
where the potential energy function,
\[ I(y) = \int_{0}^{y} f(\xi) \, d\xi. \]
two equations, the frequency parameter $\omega$ or the period, $T(=2\pi/\omega)$ is obtained as,

$$\omega = \frac{2\pi}{T} = \sqrt{1 - 2a\epsilon} \quad \ldots (5)$$

The constant, $a$ is

$$a = \frac{1 + \epsilon A - \sqrt{1 + 2\epsilon(A - 3a) - 2\epsilon^2A^2}}{3\epsilon} \quad \ldots (6)$$

Fig. 1 shows the comparison of perturbation solution and the approximate solution based on harmonic balancing technique suggested in Eq. (5) with the exact integration of the nonlinear differential Eq. (1) having the constants: $a = 1$; and $\epsilon = 0.1, 0.2$. The solution of the problem becomes singular, if the specified amplitude, $A$ is close to the saddle point, $y_2$. The asymptotic variation of the period, $T$ with the amplitude, $A$ can be seen in Fig. 1 when the amplitude, $A$ approaches the saddle point, $y_2$. It is found that the approximate solution (Eq. (5)) is close to the exact solution compared to the perturbation solution. This study demonstrates the potentiality of the method of harmonic balancing which is often criticized for not yielding consistent results.

The function $y$ in the Eq. (1) which satisfies the initial conditions (Eq. (2)), is assumed in the form

$$y = a + (A - a) \cos(\omega t) \quad \ldots (4)$$

After the use of trigonometric identities and application of the method of harmonic balance to retain only constant terms and terms involving $\cos(\omega t)$, two equations are obtained. From these

References