Magnetizable fluid behaviour with effective positive, zero or negative dynamic viscosity

Markus Zahn & Loretta L Pioch
Department of Electrical Engineering and Computer Science, Laboratory for Electromagnetic and Electronic Systems, Massachusetts Institute of Technology, Cambridge, MA 02139

Received 19 February 1998; accepted 5 November 1998

Analyses and measurements have shown anomalous behaviour of ferrofluids in ac magnetic fields, whereby in a rotating magnetic field the ferrofluid can be pumped but the flow direction can reverse as a function of magnetic field amplitude, frequency, and direction. This anomalous behaviour is investigated using the governing fluid mechanical linear and angular momentum conservation equations including non-symmetric viscous and Maxwell stress tensors. Here, a simple case has been examined where applied magnetic field components along and transverse to the duct axis are spatially uniform and vary sinusoidally with time. In the uniform magnetic field the magnetization characteristic depends on fluid spin velocity but does not depend on fluid flow velocity. The magnetization force density along the duct axis is zero while the magnetic torque density is non-zero as \( \vec{M} \) and \( \vec{H} \) are not collinear due to a magnetic relaxation time constant as well as due to spatially varying fluid spin velocity. The governing linear and angular momentum conservation equations then require non-symmetric fluid viscous and Maxwell stress tensors. Ferrofluid behaviour in ac magnetic fields offer an excellent experimental system to examine this unusual type of coupled electro-mechanical system. The governing equations are numerically integrated to solve for flow and spin velocity distributions for zero shear spin viscosity as a function of magnetic field strength, phase, frequency, and direction along and transverse to the duct axis; as a function of pressure gradient along the duct; vortex viscosity; dynamic viscosity; and ferrofluid magnetic susceptibility. Analytical solutions for simple limiting cases are given especially focussing on the case when the effective dynamic viscosity that depends on magnetic field strength can be made positive, zero or negative. Negative effective dynamic viscosity may explain the observed flow reversals while simple approximate theory for the transition point where the effective viscosity goes through zero predicts an infinite flow and spin velocity in response to a pressure gradient. The shear coefficient of spin viscosity, non-linear effects, and flow instabilities most likely limits the fluid mechanical response to remain large but finite. Numerical integrations show the highly non-linear and multi-valued solutions for flow and spin velocities when the shear spin viscosity coefficient is zero.

The motion of ferrofluid in a travelling wave magnetic field has been paradoxical as many investigators find critical magnetic field strength and frequency ranges where the fluid moves opposite to the direction of the travelling wave (backward pumping) while outside these ranges the ferrofluid moves in the same direction (forward pumping). The values of critical magnetic field strength and frequency depend on the concentration of the suspended magnetic particles and the fluid viscosity. Under ac magnetic fields, fluid viscosity acting on the magnetic particles suspended in the ferrofluid causes the magnetization \( \vec{M} \) to lag behind a travelling \( \vec{H} \). With \( \vec{M} \) not collinear with \( \vec{H} \), there is a body torque density \( \vec{T} = \mu_0 (\vec{M} \times \vec{H}) \) acting on the ferrofluid even in a uniform magnetic field even though the magnetization force density along the duct axis \( F_z = \mu_0 (\vec{M} \cdot \nabla) H_z \) is zero. Fluid mechanical analysis has been developed to extend traditional viscous fluid flows to account for the non-symmetric stress tensor that results when \( \vec{M} \) and \( \vec{H} \) are not collinear and to then simultaneously satisfy linear and angular momentum conservation for the ferrofluid.

Recent analyses on ferrofluid motion in travelling wave magnetic fields with sinusoidal time and space dependence have shown both forward and backward pumping when fluid convection and spin effects are included in the study. In that analysis, the magnetic fields were non-uniform and force and torque densities were non-zero. In this paper, ferrofluid motion is being investigated in spatially uniform sinusoidally time varying magnetic fields, both linearly polarized.
as well as travelling, where torque density is non-zero but force density along the duct axis is zero. By imposing uniform magnetic fields, it has been determined that how fluid flow is affected by different magnetic field variations including an axial component only, a transverse component only, and both axial and transverse components that are in phase for a linearly polarized field as well as magnetic field components not in phase resulting in a rotating uniform magnetic field. In the small spin velocity limit, how the effective viscosity depends on magnetic field strength and frequency so that the effective viscosity can be made larger or smaller than the fluid dynamic viscosity, including zero and negative effective viscosity has been shown. This analysis is in agreement with reported dc and low frequency measurements of increases in effective viscosity while over a range of magnetic field amplitude and frequency the effective viscosity can decrease. In addition, the small spin velocity limit is relaxed and multi-valued solutions are obtained by numerically integrating the governing non-linear differential equations with zero shear spin viscosity.

**Magnetization of Ferrofluids**

**Magnetization constitutive law**

The magnetization relaxation equation with ferrofluid undergoing simultaneous magnetization and reorientation due to fluid convection at velocity \( \vec{v} \) and particle spin at angular velocity \( \vec{\omega} \) is,\(^1\)\(^-\)\(^5\)

\[
\frac{\partial \vec{M}}{\partial t} + (\vec{v} \cdot \nabla) \vec{M} - \vec{\omega} \times \vec{M} + \frac{1}{\tau} [\vec{M} - \chi_0 \vec{H}] = 0 \tag{1}
\]

where \( \tau \) is a relaxation time constant and \( \chi_0 \) is the effective magnetic susceptibility which in general can be magnetic field dependent but in this paper will be taken to be constant. For the planar ferrofluid layer shown in Fig. 1, the flow velocity can only be \( z \)-directed and the spin velocity can only be \( y \)-directed, both quantities only varying with the \( x \) coordinate

\[
\vec{v} = v_z(x) \hat{z}, \quad \vec{\omega} = \omega_y(x) \hat{y}
\tag{2}
\]

**Magnetic fields**

Eq. (1) is being applied to the planar ferrofluid layer confined between rigid walls shown in Fig. 1. The imposed magnetic field \( H_z \) and magnetic flux density \( B_z \) are spatially uniform and are imposed on the system by external sources. Because the imposed fields are uniform with the \( y \) and \( z \) coordinates, field components can only vary with the \( x \) coordinate. Gauss's law for the magnetic flux density and Ampere's law for the magnetic field intensity with zero current density require the imposed fields to be uniform throughout the ferrofluid

\[
\nabla \cdot \vec{B} = 0 \Rightarrow \frac{dB_z}{dx} = 0 \Rightarrow B_z = \text{constant} \tag{3}
\]

\[
\nabla \times \vec{H} = 0 \Rightarrow \frac{dH_z}{dx} = 0 \Rightarrow H_z = \text{constant} \tag{4}
\]

**Fig. 1**—A planar ferrofluid layer between rigid walls is magnetically stressed by a uniform \( z \)-directed magnetic field \( H_z \) and uniform \( x \)-directed magnetic flux density \( B_x \), both of which vary sinusoidally in time at frequency \( \Omega \).

**Fig. 2**—Non-dimensional time average torque density \( \langle \vec{T}_y \rangle \) versus non-dimensional angular frequency \( \tilde{\Omega} = \Omega \tau \) for non-dimensional angular frequency \( \tilde{\Omega} = \Omega \tau = 0, 1, 5, 10 \) with \( \chi_0 = 1 \) and with (a) axial magnetic field \( (\tilde{H}_z = H_0, \tilde{B}_x = \mu_0 H_0) \); (b) transverse magnetic field \( (\tilde{H}_z = 0, \tilde{B}_x = \mu_0 H_0) \); (c) linearly polarized magnetic field \( (\tilde{H}_z = H_0, \tilde{B}_x = 0) \); and (d) rotating magnetic field \( (\tilde{H}_z = j H_0, \tilde{B}_x = \mu_0 H_0) \).
There are no sources for $y$ components of magnetic field. Therefore, the total magnetic field $\vec{H}$ and magnetic flux density $\vec{B}$ inside the ferrofluid layer vary sinusoidally in time and are of the form,

$$\vec{B} = \mathcal{R} \{ [\vec{B}_x \hat{i}_x + \vec{B}_z(x) \hat{i}_z] e^{j\Omega t} \} \quad \ldots (5)$$

$$\vec{H} = \mathcal{R} \{ [\vec{H}_x(x) \hat{i}_x + \vec{H}_z \hat{i}_z] e^{j\Omega t} \} \quad \ldots (6)$$

where $j = \sqrt{-1}$ and

$$\vec{B} = \mu_0(\vec{H} + \vec{M}). \quad \ldots (7)$$

$\vec{M}_x$ and $\vec{M}_z$ are needed to be solved in terms of the known imposed field amplitudes $\vec{H}_z$ and $\vec{B}_x$. Substituting Eqs (5) and (6) into Eq. (1) relates the magnetization components to the magnetic field $\vec{H}$ as,

$$j\Omega \vec{M}_x - \omega_y \vec{M}_x + \frac{\vec{M}_x}{r} = \frac{\chi_0}{\mu_0} \vec{H}_x \quad \ldots (8)$$

$$j\Omega \vec{M}_z - \omega_y \vec{M}_z + \frac{\vec{M}_z}{r} = \frac{\chi_0}{\mu_0} \vec{H}_z \quad \ldots (9)$$

where the second term in Eq. (1) has zero contribution because $\vec{v}$ is only $z$-directed and $\vec{M}$ can only vary with $x$. By taking the $x$-component of Eq. (7) and solving for $\vec{H}_x$, Eq. (10) is obtained,

$$\vec{B}_x = \mu_0(\vec{H}_x + \vec{M}_x) \Rightarrow \vec{H}_x = \frac{\vec{B}_x}{\mu_0} - \vec{M}_x \quad \ldots (10)$$

By using Eq. (10) in Eq. (8)

$$\vec{M}_x = \frac{\chi_0}{[(\omega_y \tau)^2 + (j\Omega \tau + 1)(j\Omega \tau + 1 + \chi_0)]} \left[ (\omega_y \tau) \vec{H}_x + \frac{\vec{B}_x}{\mu_0} \right] \quad \ldots (11)$$

$$\vec{M}_z = \frac{\chi_0}{[(\omega_y \tau)^2 + (j\Omega \tau + 1)(j\Omega \tau + 1 + \chi_0)]} (j\Omega \tau + 1 + \chi_0) \vec{H}_z - \frac{\vec{B}_x \omega_y \tau}{\mu_0} \quad \ldots (12)$$

Eqs (11) and (12) represent the magnetization of the ferrofluid as a function of the imposed fields $\vec{H}_z$ and $\vec{B}_x$ and as a function of the not yet known spin velocity $\omega_s$ which can vary with position $x$. The magnetization gives rise to a torque on the ferrofluid which causes fluid motion and thus a non-zero $\omega_s$. The resulting $\omega_s$ then changes the magnetization. There is a strong magneto-mechanical coupling so that the magnetization and mechanical equations need to be self-consistently satisfied.

Fluid mechanics

For incompressible fluids,

$$\nabla \vec{v} = 0; \ \nabla \cdot \vec{\omega} = 0 \quad \ldots (13)$$

and the coupled linear and angular momentum conservation equations for force density $\vec{f}$ and torque density $\vec{T}$ for a fluid in a gravity field $-g\hat{i}_x$ are

$$\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\nabla p + \vec{f} + 2\zeta \nabla \times \vec{\omega} \quad \ldots (14)$$

$$+ (\zeta + \eta) \nabla^2 \vec{v} - \rho g \hat{i}_x$$

$$I \left[ \frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \nabla) \vec{\omega} \right] = \vec{T} + 2\zeta (\nabla \times \vec{v} - 2\vec{\omega}) + \eta' \nabla^2 \vec{\omega} \quad \ldots (15)$$

where $\rho$ is the mass density, $p$ is the pressure, $\zeta$ is the vortex viscosity, $\eta$ is the dynamic viscosity, $I$ is the moment of inertia density, and $\eta'$ is the shear coefficient of spin-viscosity.

These equations can be applied to the planar ferrofluid layer confined between rigid walls (Fig. 1). It is assumed that the planar ferrofluid has viscous dominated flow so that inertia is negligible and is in the steady state so that the fluid responds only to the time average magnetic force and torque densities.
Magnetic force and torque densities

Magnetic force density—For \(0 \leq x < d\), the magnetic force density is given by,

\[
\mathbf{f} = \mu_0 \left( \mathbf{M} \cdot \nabla \right) \mathbf{H}
\]

Solving for the \(x\) and \(z\) components with field components only constant or varying with \(x\),

\[
f_x = \mu_0 M_x \frac{dH_x}{dx} = \mu_0 M_x \left( \frac{1}{\mu_0} M_x - M_x \right)
\]

\[
= -\mu_0 M_x \frac{dM_x}{dx} = -\frac{d}{dx} \left( \frac{1}{2} \mu_0 M_x^2 \right)
\]

\[
f_z = \mu_0 M_z \frac{dH_z}{dx} = 0
\]

The time average components of the magnetic force density are then,

\[
\langle f_x \rangle = -\frac{d}{dx} \left( \frac{1}{4} \mu_0 |M_x|^2 \right)
\]

\[
\langle f_z \rangle = 0
\]

Magnetic torque density—Similarly, the torque density is given by,

\[
\mathbf{T} = \mu_0 \left( \mathbf{M} \times \mathbf{H} \right) = \mu_0 \left( -M_x H_z + M_z H_x \right) \mathbf{I}_y
\]

The torque is only \(y\)-directed,

\[
T_y = -\mu_0 M_x H_z + \mu_0 M_z \left( \frac{B_x}{\mu_0} - M_x \right)
\]

\[
= M_z B_x - \mu_0 M_x \left( H_z + M_z \right)
\]

The time average component of torque density is then,

\[
\langle T_y \rangle = \frac{1}{2} \left[ \dot{M}_z \dot{B}_x^* - \mu_0 \dot{M}_x^* \left( \dot{H}_z + \dot{M}_z \right) \right]
\]

with superscript asterisks (*) indicating the complex conjugate of a complex amplitude field quantity. In Fig. 2, non-dimensional torque density \(\langle T_y \rangle / (\mu_0 H_0^2)\) is plotted as a function of non-dimensional spin velocity \(\omega_y = \omega_y \tau \) with \(H_0\) a nominal magnetic field amplitude. Four limiting cases with the magnetic field purely axially directed \((\dot{H}_z = H_0, \dot{B}_x = 0)\), purely transversely directed \((\dot{H}_z = 0, \dot{B}_x = \mu_0 H_0)\) magnetic field linearly polarized with \(\dot{H}_z = H_0\) and \(\dot{B}_x = \mu_0 H_0\) non-zero and in phase, and a rotating magnetic field with \(\dot{H}_z = jH_0\) and \(\dot{B}_x = \mu_0 H_0\) having a \(\pi/2\) phase difference. Throughout this paper, non-dimensional variables are indicated with a tilde. In Fig. 2, it may be noted that for a stationary spin velocity \((\omega_y = 0)\) that the slope around \(\omega_y = 0\) is negative at low frequencies and positive at high frequencies and that both axial and transverse magnetic field components must be non-zero to have non-zero time average torque at \(\omega_y = 0\).

Coupled linear and angular momentum conservation equations—it is convenient to define a modified pressure as,

\[
p' = p + \frac{1}{4} \mu_0 |\dot{M}_x|^2 + \rho \omega_y \tau \frac{\partial p'}{\partial \tau} = 0
\]

whose variation with \(x\) is zero from the \(x\) component of Eq. (14) with \(\nu_x = 0\). Then Eqs (14) and (15) in the negligible inertia, viscous dominated limit become,

\[
\left( \zeta + \eta \right) \frac{d^2 \nu_x}{dx^2} + 2\zeta \frac{d \omega_y}{dx} - \frac{\partial p'}{\partial z} = 0
\]

\[
\eta' \frac{d^2 \omega_y}{dx^2} - 2\zeta \left( \frac{d \nu_x}{dx} + 2\omega_y \right) + \langle T_y \rangle = 0.
\]

Normalized general equations

It is convenient to express parameters in non-dimensional form indicated with tildes, with time normalized to the magnetic relaxation time \(\tau\), space normalized to duct spacing \(d\), and magnetic field quantities normalized to a nominal magnetic field strength \(H_0\),

\[
\tilde{\Omega} = \Omega \tau; \quad \tilde{H} = \frac{\dot{H}}{H_0}; \quad \tilde{B} = \frac{\dot{B}}{\mu_0 H_0}; \quad \tilde{M} = \frac{\dot{M}}{H_0};
\]

\[
\tilde{x} = \frac{x}{d}; \quad \tilde{v}_x = \frac{v_x}{d}; \quad \tilde{\omega}_x = \omega_x \tau; \quad \tilde{v}_y = \frac{v_y}{d}; \quad \tilde{\omega}_y = \omega_y \tau;
\]

\[
\tilde{\eta}' = \frac{2\eta'}{\mu_0 H_0^2 \tau}; \quad \tilde{\eta} = \frac{\eta' d^2}{\mu_0 H_0^2 \tau}; \quad \zeta = \frac{2\zeta}{\mu_0 H_0^2 \tau};
\]

\[
\frac{\partial p'}{\partial z} = \frac{d}{\mu_0 H_0^2} \frac{\partial p'}{\partial z}.
\]
Then the, non-dimensional flow and spin velocity equations are,

\[ \frac{1}{2} (\zeta + \eta) \frac{d^2 \tilde{v}_x}{d \tilde{x}^2} + \zeta \frac{d \tilde{\omega}_y}{d \tilde{z}} - \frac{\partial \tilde{\omega}_x}{\partial \tilde{z}} = 0 \quad \ldots (28) \]

\[ \tilde{v}_x \frac{d^2 \tilde{\omega}_y}{d \tilde{x}^2} - \zeta \left( \frac{d \tilde{v}_y}{d \tilde{x}} + 2 \tilde{\omega}_y \right) + \langle \tilde{T}_y \rangle = 0 \quad \ldots (29) \]

where

\[ \langle \tilde{T}_y \rangle = \frac{1}{2} \Re \left[ \tilde{M}_y \tilde{B}_x^* - \tilde{M}_x \tilde{H}_x + \tilde{M}_z \right] \quad \ldots (30) \]

and

\[ \tilde{M}_x = \frac{\chi_0 \left[ (\tilde{\omega}_y \tilde{H}_x + (\tilde{\omega}_x + \chi_0) \tilde{B}_x \right]}{\tilde{\omega}_y + (\tilde{\omega}_y + \chi_0) \tilde{\omega}_y} \quad \ldots (31) \]

\[ \tilde{M}_z = \frac{\chi_0 \left[ (\tilde{\omega}_y + \chi_0) \tilde{H}_x - \tilde{B}_x \tilde{\omega}_y \right]}{\tilde{\omega}_y + (\tilde{\omega}_y + \chi_0) \tilde{\omega}_y} \quad \ldots (32) \]

This set of equations describes the motion of the planar ferrofluid layer confined between rigid walls with imposed spatially uniform, sinusoidally time varying \( x \) and \( z \) directed magnetic fields. The primary complexity of the analysis is that the time average torque density \( \langle \tilde{T}_y \rangle \) in Eq. (30) depends in a complicated way on the spin velocity \( \tilde{\omega}_y \) which depends on \( \tilde{x} \).

Substituting Eqs (31)-(32) into Eq. (30) gives,

\[ \langle \tilde{T}_y \rangle = \frac{\chi_0}{2} \left[ \tilde{\omega}_y \left[ (\tilde{B}_x \tilde{H}_x - \tilde{\omega}_y \tilde{H}_x + (\tilde{\omega}_y + 1) \tilde{H}_x \right] \right]^2 \]

\[ + 29 \Re \left[ \chi_0 (\tilde{\omega}_y^2 - \tilde{\omega}_y - 1 - \chi_0) \right] \]

\[ + (2 + \chi_0)^2 \tilde{\omega}_y^2 \quad \ldots (33) \]

It may be noted that the phase relationship between \( \tilde{H}_x \) and \( \tilde{B}_x \) is very important in the third term of the numerator. With \( \eta' \neq 0, \tilde{\omega}_y \) must be zero at the fixed boundaries at \( x=0 \) and \( x=d \), and it is useful to examine Eq. (33) in the limit of small \( \tilde{\omega}_y \). To first order in \( \tilde{\omega}_y \), Eq. (33) approximately reduces to,

\[ \lim_{\tilde{\omega}_y \ll 1} \langle \tilde{T}_y \rangle \approx \tilde{T}_0 + \alpha \tilde{\omega}_y \quad \ldots (34) \]

where

\[ \tilde{T}_0 = -\frac{\chi_0^3 \Re \left[ \tilde{H}_x \tilde{H}_x^* + \tilde{\omega}_y \tilde{H}_x \tilde{H}_x^* \right][\tilde{H}_x \tilde{B}_x^*]}{[1 + \chi_0 + \tilde{\omega}_y^2]^2 + \chi_0^2 \tilde{\omega}_y^2} \quad \ldots (35) \]

\[ \alpha = \frac{\chi_0}{2} \left[ \tilde{B}_x^2 \tilde{H}_x^2 - \tilde{H}_x^2 \tilde{B}_x^2 \right] \quad \ldots (36) \]

It may be noted in particular that as shown in Fig. 2 for purely axial \( (\tilde{B}_x = 0) \) or transverse \( (\tilde{H}_x = 0) \) magnetic fields that \( \tilde{T}_0 = 0 \). Again it may be noted that \( \alpha \) can be positive or negative depending on the value of \( \Omega \) compared to 1 or \( 1 + \chi_0 \); \( \alpha \) is positive at high frequencies and negative at low frequencies.

### Zero Spin Viscosity (\( \eta' = 0 \)) Solution Methods

#### General solutions

Simple limiting case of zero spin viscosity \( (\eta' = 0) \) has been examined. Eq. (29) has been differentiated with respect to \( \tilde{x} \), and solved for

\[ \frac{d \tilde{\omega}_y}{d \tilde{x}} \]

\[ \frac{d \tilde{\omega}_y}{d \tilde{x}} = -\frac{1}{2} \frac{d^2 \tilde{v}_y}{d \tilde{x}^2} + \frac{d \tilde{v}_y}{d \tilde{x}} \quad \ldots (37) \]

Substituting this into Eq. (28) and solving for \( \frac{d^2 \tilde{v}_y}{d \tilde{x}^2} \) yields

\[ \frac{d^2 \tilde{v}_y}{d \tilde{x}^2} = -\frac{2 \partial \tilde{p}'}{\tilde{\eta} \partial \tilde{z}} - \frac{1}{\tilde{\eta}} \frac{d \langle \tilde{T}_y \rangle}{d \tilde{x}} \quad \ldots (38) \]

Integrating Eq. (38) twice, one obtains

\[ \tilde{v}_z (\tilde{x}) = \frac{1}{\tilde{\eta}} \frac{\partial \tilde{p}'}{\partial \tilde{z}} \tilde{x}^2 - \frac{1}{\tilde{\eta}} \int_0^{\tilde{x}} < \tilde{T}_y > d \tilde{x} + k_1 \tilde{x} + k_2 \quad \ldots (39) \]

where \( k_1 \) and \( k_2 \) are constants of integration to be found from the boundary conditions of zero velocity at the rigid boundaries

\[ \tilde{v}_z (\tilde{x} = 0) = 0; \quad \tilde{v}_z (\tilde{x} = 1) = 0 \quad \ldots (40) \]
Fig. 4—Representative non-linear and multi-valued spin velocity spatial distributions for various positive, zero, and negative values of $\tilde{\eta}_{\text{eff}}$ in a magnetic field transverse to the duct axis at frequency $\tilde{\Omega}=10$. Spin velocity profiles for $\zeta$ values to the (a) right and (b) left of the $\tilde{\eta}_{\text{eff}}=0$ curve in Fig. 3.

Applying these boundary conditions to Eq. (39),

$$k_1 = -\frac{1}{\tilde{\eta}} \frac{\partial \tilde{p}'}{\partial \tilde{z}} + \frac{1}{\tilde{\eta}} \int_0^1 <\tilde{T}_y> d\tilde{x} : k_2 = 0 \quad (41)$$

so that the velocity is

$$\tilde{v}_z(\tilde{x}) = \frac{1}{\tilde{\eta}} \frac{\partial \tilde{p}'}{\partial \tilde{z}} (\tilde{x} - 1) + \frac{1}{\tilde{\eta}} \int_0^1 <\tilde{T}_y> d\tilde{x}$$

$$- \int_0^1 <\tilde{T}_y> d\tilde{x}$$

$$\quad (42)$$

We solve for $\tilde{\omega}_y(\tilde{x})$ in Eq. (37) as

$$\tilde{\omega}_y = \frac{<\tilde{T}_y>}{2\tilde{\zeta}} \frac{1}{2} \frac{d\tilde{v}_z}{d\tilde{x}}$$

$$= -\frac{1}{2\tilde{\eta}} \left[ \frac{d\tilde{p}'}{d\tilde{z}} (2\tilde{x} - 1) - \frac{\tilde{v}_z}{\zeta} <\tilde{T}_y> + \int_0^1 <\tilde{T}_y> d\tilde{x} \right]$$

$$\quad (43)$$

It is noted that Eqs (42) and (43) are not known analytical solutions for $\tilde{v}_z$ and $\tilde{\omega}_y$ because $<\tilde{T}_y>$ varies with $\tilde{x}$ because $\tilde{\omega}_y$ varies with $\tilde{x}$. However Eqs (42) and (43) do help explain key features of ferrofluid flow for $\tilde{\eta}'=0$.

When $\partial \tilde{p}'/\partial \tilde{z}$ is larger compared to the torque density $<\tilde{T}_y>$, then the flow velocity $\tilde{v}_z$ will be essentially parabolic being maximum at $\tilde{x}=0.5$ which is the usual Poiseuille flow in a planar duct, while the spin velocity $\tilde{\omega}_y$ will be linear with respect to $\tilde{x}$ being zero at the midpoint $\tilde{x}=0.5$.

If the time average torque density is constant with position, then the torque has no contribution to the flow velocity as the two torque terms in Eq. (42) cancel but there is a constant torque contribution to the spin velocity in Eq. (43). If there is no pressure gradient, $\partial \tilde{p}'/\partial \tilde{z} = 0$, then the solution for $\tilde{\omega}_y$ in Eq. (43) is a constant, independent of $\tilde{x}$ as $<\tilde{T}_y>$ is only constant given by Eq. (33). Then the solution to Eq. (42) is $\tilde{v}_z(\tilde{x})=0$ and $\tilde{\omega}_y(\tilde{x})=\frac{<\tilde{T}_y>}{2\tilde{\zeta}}$. The torque is constant with position if $\tilde{\omega}_y$ is constant with position as given by Eq. (33).

Small spin velocity limit

If the torque is linear with position, it contributes parabolically to the flow velocity profile and linearly to the spin velocity profile in exactly the same way as the pressure gradient so that the net effective viscosity is magnetic field dependent. This occurs in the small spin velocity
Fig. 5—Representative non-linear and multi-valued flow velocity spatial distributions for various positive, zero, and negative values of \( \tilde{\eta}_{\text{eff}} \) in a magnetic field transverse to the duct axis at frequency \( \tilde{\Omega} = 10 \). Flow velocity profiles for \( \zeta \) values to the (a) right and (b) left of the \( \tilde{\eta}_{\text{eff}} = 0 \) curve in Fig. 3.

When Eq. (34) is valid. Substituting Eq. (34) into Eqs (42)-(43), results in the approximate flow and spin velocity profiles,

\[
\tilde{\nu}_z(\tilde{x}) = \frac{\tilde{x}(\tilde{x} - 1) \partial \tilde{p}'}{\tilde{\eta}_{\text{eff}} \partial \tilde{z}} \quad \ldots \quad (44)
\]

\[
\tilde{\omega}_y(\tilde{x}) = \frac{1}{(2\tilde{\zeta} - \alpha)} \left[ \tilde{T}_0 - \frac{\tilde{\zeta} (2\tilde{x} - 1) \partial \tilde{p}'}{\tilde{\eta}_{\text{eff}} \partial \tilde{z}} \right] \quad \ldots \quad (45)
\]

where the effective viscosity is

\[
\tilde{\eta}_{\text{eff}} = \tilde{\eta} - \frac{\alpha \tilde{\zeta}}{2\tilde{\zeta} - \alpha} \quad \ldots \quad (46)
\]

To compare to previously published results of the effective viscosity, \( \alpha < 2\tilde{\zeta} \) and \( \chi_0 < 1 \) is considered to that the change in viscosity is

\[
\Delta \tilde{\eta} = -\frac{\alpha}{2} = \frac{\chi_0 (1 - \tilde{\Omega}^2) [1 \tilde{B}_y^2 + \tilde{B}_z^2]}{2(1 + \tilde{\Omega}^2)^2} \quad \ldots \quad (47)
\]

This agrees with the results reported earlier in the weak field Langevin limit treated there. It may be noted that the effective viscosity is decreased for \( \tilde{\Omega} > 1 \) and increased for \( \tilde{\Omega} < 1 \). Referring back to the general result of Eq. (46), the effective viscosity is zero when

\[
\alpha = \frac{2\tilde{\zeta}}{\tilde{\zeta} + \tilde{\eta}} \Rightarrow \tilde{\eta}_{\text{eff}} = 0 \quad \ldots \quad (48)
\]

but then Eq. (45) shows that \( \tilde{\omega}_y \) becomes infinite, violating the small spin velocity approximation of Eq. (34). For \( \Delta \tilde{\eta} < -\tilde{\eta} \), the effective viscosity becomes negative. Could this be the cause for flow reversal reported with travelling magnetic fields? It may also be noted in Eq. (45) that the singularity for \( \tilde{T}_0 \neq 0 \) when \( \alpha = 2\tilde{\zeta} \), so that \( \tilde{\omega}_y(\tilde{x}) \) becomes infinite, but again violating the small spin velocity approximation.

Governing non-linear differential equations

With \( \tilde{\eta}' = 0 \), Eq. (29) can be rewritten as,

\[
\frac{d\tilde{\nu}_z}{d\tilde{x}} = \frac{\tilde{T}_y}{\tilde{\zeta}} - 2\tilde{\omega}_y \quad \ldots \quad (49)
\]

Then substituting into Eq. (28), and recognizing that,

\[
\frac{d<\tilde{T}_y>}{d\tilde{x}} = \frac{d<\tilde{T}_y>}{d\tilde{\omega}_y} \frac{d\tilde{\omega}_y}{d\tilde{x}} \quad \ldots \quad (50)
\]
IB = 0, IH = 1 for given $\eta_{el}$

$\eta = \bar{\eta}$ for different than the dynamic viscosity $\bar{\eta}$ of the carrier fluid. Here, the special case of a magnetic field is considered purely transverse to the duct axis so that, $|B_x| = 1, |H_z| = 0$. To further simplify the algebraic analysis the special case is considered when the vortex and dynamic viscosities are equal, $\bar{\eta} = \bar{\eta}$. Then solving Eq. (46) for the parameter $\alpha$ of Eq. (36) in terms of $\bar{\eta} = \bar{\eta}$ and $\bar{\eta}_{eff}$ yields,

$$\alpha = \frac{2\bar{\eta}(\zeta - \bar{\eta}_{eff})}{2\bar{\eta} - \bar{\eta}_{eff}} = \frac{\chi_0}{2(1 + \chi_0 + \bar{\Omega}_x^2) + \chi_0^2\bar{\Omega}_z^2}$$

Then solving Eqs (54) for $\bar{\Omega}$ results in fourth order biquadratic equation,

$$\bar{\Omega}^4 + \bar{\Omega}_x^2 \left[ (1 + \chi_0)^2 + 1 - \frac{\chi_0}{2\alpha} \right] + \frac{(1 + \chi_0)^2 + \chi_0}{2\alpha} = 0$$

with solutions

$$\bar{\Omega} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

where

$$b = \left[ (1 + \chi_0)^2 + 1 - \frac{\chi_0}{2\alpha} \right]$$

$$c = \left[ (1 + \chi_0)^2 + \frac{\chi_0}{2\alpha} \right]$$

The frequency $\bar{\Omega}$ versus $\zeta$ for various values of positive, zero and negative $\bar{\eta}_{eff}$ is plotted in Fig. 3 for cases with $\bar{\Omega}$ real and positive. Solutions for negative $\bar{\eta}_{eff}$ fall in a region between the positive viscosity solutions in the range of $0 \leq \bar{\eta}_{eff} \leq \infty$. Note that for a given $\bar{\Omega}$ and positive $\bar{\eta}_{eff}$ that there are two possible values of $\zeta$ while for a negative $\bar{\eta}_{eff}$ there is only one possible $\zeta$.

Once values of $\zeta$ and $\bar{\eta}_{eff}$ are chosen, the value of frequency $\bar{\Omega}$ is determined from Fig. 3. To
Fig. 7—Representative non-linear and multi-valued spin velocity spatial distributions for various positive, zero, and negative values of $\tilde{\eta}_{\text{eff}}$ in a magnetic field parallel to the duct axis at frequency $\tilde{\Omega}=10$. Spin velocity profiles for $\tilde{\zeta}$ values to the (a) right and (b) left of the $\tilde{\eta}_{\text{eff}}=0$ curve in Fig. 6.

determine the spin velocity spatial distribution by numerically integrating Eq. (51), it is useful to realize that for the transverse magnetic field case that $\tilde{\phi}_y$ is an odd function of $\tilde{x}=0.5$ so that $\tilde{\phi}_y(\tilde{x}=0.5)=0$ while $\tilde{\nu}_z$ is an even function of $\tilde{x}=0.5$. Then Eq. (51) can be solved for spatial distributions of spin velocity by Runge-Kutta numerical integration from $\tilde{\phi}_y(\tilde{x}=0.5)=0$ to yield non-linear and multi-valued solutions near $\tilde{x}=0.5$ as shown in Fig. 4. For all plots in this paper numerical values of $\chi_0=1$ and $\partial \tilde{p}/\partial \tilde{z}=1$ are used. In Fig. 4a spatial profiles of $\tilde{\phi}_y$ for $\tilde{\Omega}=10$ for positive values of $\tilde{\eta}_{\text{eff}}$ with $\tilde{\zeta}$ values to the right of the $\tilde{\eta}_{\text{eff}}=0$ curve in Fig. 3 are shown. In Fig. 4b spatial profiles are shown for positive and negative values of $\tilde{\eta}_{\text{eff}}$ with $\tilde{\zeta}$ values to the left of the $\tilde{\eta}_{\text{eff}}=0$ curve in Fig. 3.

The flow velocity distribution, $\tilde{\nu}_z$ can then be numerically determined from Eq. (53) as a function of $\tilde{\phi}_y$ and parametrically plotted as a function of $\tilde{x}$. Representative plots are shown in Fig. 5 for various positive, zero, and negative values of $\tilde{\eta}_{\text{eff}}$ to also have multi-valued solutions near $\tilde{x}=0.5$. The essentially parabolic like profile is similar to that of usual Poiseuille pipe flow, but with magnetic field dependent effective viscosity. It may be noted that the multi-valued non-linear behaviour at the three locations centered at $\tilde{x}=0.5$ are easily seen for $\tilde{\Omega}$ and $\tilde{\eta}_{\text{eff}}$ solutions to the left of the $\tilde{\eta}_{\text{eff}}=0$ curve in Fig. 3 and less easily seen for curves to the right of the $\tilde{\eta}_{\text{eff}}=0$ curve in Fig. 3. These multi-valued solutions are probably non-physical due to the assumption that $\eta'=0$ and due to flow instabilities. However, they do point to the possibility of unusual flow behaviour.

Axial magnetic field

Here it is considered that the special case of a magnetic field purely parallel to the duct axis so that $|\tilde{B}_x|=0, |\tilde{H}_z|=1$. The algebraic analysis is continued to be simplified by assuming the special case when the vortex and dynamic viscosities are equal $\tilde{\eta}=\tilde{\zeta}$. Then solving Eq. (46) for the parameter $\alpha$ of Eq. (36) in terms of $\tilde{\zeta}$ and $\tilde{\eta}_{\text{eff}}$ yields

$$\alpha = \frac{2\tilde{\zeta}(\tilde{\zeta}-\tilde{\eta}_{\text{eff}})}{2\tilde{\zeta}-\tilde{\eta}_{\text{eff}}} = \frac{x_0}{2} \left[ \frac{\tilde{\Omega}^2-(1+x_0)^2}{[1+x_0+\tilde{\Omega}^2+x_0^2\tilde{\Omega}^2]} \right]$$

... (58)

Then solving Eq. (59) for $\tilde{\Omega}$ results in a fourth order biquadratic equation,

$$\tilde{\Omega}^4 + \tilde{\Omega}^2 \left[ (1+x_0)^2 + 1 - \frac{x_0}{2\alpha} \right] + (x_0+1)^2 \left( 1 + \frac{x_0}{2\alpha} \right) = 0$$

... (59)
Fig. 8—Representative non-linear and multi-valued flow velocity spatial distributions for various positive, zero, and negative values of $\eta_{\text{eff}}$ in a magnetic field transverse to the duct axis at frequency $\tilde{\Omega} = 10$. Flow velocity profiles for $\zeta$ values to the (a) right and (b) left of the $\tilde{\eta}_{\text{eff}} = 0$ curve in Fig. 6.

with solutions given as Eq. (56) with

\[
b = \left[ (1 + x_0)^2 + 1 - \frac{x_0}{2\alpha} \right] \quad \ldots \quad (60)
\]

\[
c = \left( x_0 + 1 \right) \left( 1 + \frac{x_0}{2\alpha} \right)
\]

The frequency $\tilde{\Omega}$ versus $\zeta$ for various values of $\tilde{\eta}_{\text{eff}}$ is plotted in Fig. 6 and is similar to form to the curves in Fig. 3 for the transverse magnetic field case.

Once values of $\zeta$ and $\tilde{\eta}_{\text{eff}}$ are chosen, the value of frequency $\tilde{\Omega}$ is determined. To determine the spin velocity spatial distribution by numerically integrating Eq. (51), it is again true that for the axial magnetic field case that $\tilde{\omega}_y$ is an odd function of $\tilde{x}$ while $\tilde{\omega}_z$ is an even function of $\tilde{x}$. Following the same procedure as mentioned above the spatial distributions of spin and linear velocities are found by Runge-Kutta numerical integration to again yield non-linear and multi-valued solutions near $\tilde{x} = 0.5$ as shown in Figs 7 and 8.

**Linearly polarized and rotating uniform magnetic fields**

Other interesting special cases to be examined in the future are when $\tilde{B}_x$ and $\tilde{H}_z$ are both non-zero. Linearly polarized magnetic fields occur when these two field components are in phase, while rotating fields result if these two field components are $90^\circ$ out of phase, for example if one component is pure real and the other is pure imaginary. As two special cases, consider $\tilde{B}_x = 1$ and $\tilde{H}_z = 1$ for linearly polarized magnetic fields and $\tilde{B}_x = 1$ and $\tilde{H}_z = j$, for a rotating magnetic field. For both cases, $|\tilde{B}_x| = |\tilde{H}_z| = 1$ so that from Eqs (36) and (46),
the spin and linear velocities are no longer respectively odd and even distributions in space. Thus, more intensive numerical work is necessary to solve for the spin velocities as \( \omega_y(\chi = 0.5) \neq 0 \) so that no initial value of \( \omega_y \) is known to allow Runge-Kutta numerical integration of Eq. (51) to begin.

Acknowledgements
This work has been partially supported by a grant from the Exxon Education Foundation’s Research and Training Program. Stimulating conversations with Dr R.E. Rosensweig are greatly appreciated.

References