The closed-form solutions for a nonsimilar viscoelastic boundary layer flow and heat transfer

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The nonsimilar boundary layer equations for the flow of an incompressible second-order fluid over a stretching sheet are solved analytically to obtain the coefficients of skin-friction and heat transfer.

With increasing prospects for non-Newtonian fluids due to their extensive use in polymer industry, there is a renewed interest in the study of viscoelastic boundary layer phenomena. Amongst the many models, the Coleman-Noll constitutive equation,

\[ T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \]  

(1)

which is based on the postulate of gradually fading memory for an incompressible second-order fluids, has received special attention. Here \( T \) is the stress tensor, \( p \) is the pressure, \( \mu, \alpha_1, \alpha_2 \) are material constants with \( \alpha_1 < 0 \) (which follows from thermodynamic considerations) and \( A_1, A_2 \) are defined by

\[ A_1 = (\nabla \vec{V}) + (\nabla \vec{V})^T \]  

(2)

\[ A_2 = \frac{dA_1}{dt} + A_1(\nabla \vec{V}) + (\nabla \vec{V})^T \cdot A_1 \]  

(3)

where \( \vec{V} \) denotes velocity. The model (1) displays normal-stress differences in shear flow and is an approximation to a simple fluid in the sense of retardation. This model is applicable to some dilute polymer solutions and is valid at low rates of shear. Dilute polymer solutions like 0.83\% ammonium alginate in water and 5.4\% polyisobutylene in cetane have approximate Prandtl number of 440 and 3 respectively.

The objective of this study is to obtain analytical solution for the problem of heat transfer in a viscoelastic fluid obeying Eq. (1) past a semi-infinite stretching sheet considering various aspects such as the effects of wall temperature, frictional heating, internal heat generation or absorption and magnetic field. The heat transfer co-efficient is also obtained for very large Prandtl numbers.

Analysis—Heat transfer in the steady laminar flow of an incompressible second-order fluid obeying Eq. (1) past a semi-infinite stretching sheet is considered. The fluid is at rest and the motion is created by the stretching of the sheet with velocity, \( u_e = cx, \ c > 0 \)...

The difference of surface temperature \( T \) and that of ambient fluid \( T_a \) is assumed as

\[ T_e - T = A - \omega \]  

where \( A \) and \( \omega \) are constants, and \( L \) is the characteristic length. Low magnetic Reynolds number flow is assumed so that the externally applied transverse uniform magnetic field, \( B_y \) is undisturbed by the flow. Under the foregoing assumptions, the boundary layer equations for an incompressible viscoelastic fluid flow can be written in the form

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  

(6)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - k \left( \frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial y^2} \right) + \nu \frac{\partial^3 u}{\partial y^3} - \frac{\sigma B_y^2 u}{\rho} \]  

(7)

\[ \rho C_p \left( \frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial y} \right) = \frac{k}{\rho} \left( \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \right) + Q(T - T_a) + \sigma B_y^2 u^2 \]  

(8)

The boundary conditions are

\[ u = u_e, \quad v = 0, \quad T = T_w \quad \text{at} \quad y = 0 \]  

(9)

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In the foregoing equations the imposed electric field is absent and the electric field due to charge polarization is assumed to be negligible. Here $u$ and $v$ are the velocity components along $x$ and $y$ directions respectively. $x$ and $y$ are the distances along and perpendicular to the surface. $Q$ is the specific heat generation rate. $T$ is the temperature $k = -(a_1/\rho)$ is a positive parameter associated with the viscoelastic fluid. $K$ is the thermal conductivity. $p$ is the density, $\nu = (\mu/\rho)$ is the kinematic viscosity.$C_p$ is the specific heat at constant pressure and $a$ is the electric conductivity.

Introducing the stream function $\psi$ as
\[ u = \frac{\partial \psi}{\partial \gamma} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \quad \ldots \quad (11) \]
and defining
\[ \bar{\psi} = \frac{\psi}{u_0 L} = \frac{x}{\sqrt{Re}} f(\eta), \quad \theta = \frac{T-T_\infty}{T_w-T_\infty} \quad \ldots \quad (12) \]

Eqs (6)-(10) are transformed to
\[ (f')^2 - f'' = f' - k_2 (2f f' - f' f') \quad \ldots \quad (13) \]
\[ \bar{x} \partial \theta / \partial x + \beta f' \theta - f' = \frac{1}{Pr} \theta'' + \alpha \theta + Ec \bar{x}^2 ((f')^2 - k_1 (f' f' - f'' f') + I (f' f')) \quad \ldots \quad (14) \]
\[ f = 0, f' = 1, \theta = 1 \quad \text{at} \quad \eta = 0 \quad \ldots \quad (15) \]
\[ f' \rightarrow 0, \theta \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty \quad \ldots \quad (16) \]
where $\bar{x} = \frac{x}{L}, \bar{y} = \frac{y}{L}, u_0 = c L, k_1 = \frac{k c}{v}, \eta = \sqrt{Re} \bar{y}$,
\[ Re = \frac{u_0 L}{v} \quad \text{is the Reynolds number}, \quad Ec = \frac{u_0^2}{C_p (T_w - T_\infty)} \quad \text{is the Eckert number}, \quad Pr = \frac{\mu C_p}{\kappa} \quad \text{is the Prandtl number}, \quad \alpha = \frac{Q}{\rho C_p} \quad \text{is the heat source/sink parameter}, \quad I = \frac{\sigma B^2 L}{\rho u_0} \quad \text{is the magnetic interaction parameter}, \quad L \quad \text{and} \quad u_0 \quad \text{are characteristic length and velocity respectively, and primes denote differentiation with respect to} \ \eta. \ \text{The momentum Eq. (13) is uncoupled from the energy Eq. (14). The transformation coordinate normal to the surface} \ \eta \ \text{indicates that the order of the momentum boundary layer thickness is} \ \eta \rightarrow 0, \ T \rightarrow T_\infty \ \text{as} \ \eta \rightarrow \infty \quad \ldots \quad (10) \]

The skin-friction co-efficient, $C_f$ can be expressed as
\[ C_f = \frac{\tau_y |_{\eta=0}}{\frac{1}{2} \rho u_0^2} = \frac{2}{\sqrt{Re \cdot \bar{x}}} (1 - 3 k_1) f''(0) \quad \ldots \quad (17) \]

where $\tau_y$ is the shear stress.

The local heat transfer co-efficient (Nusselt number) can be written as
\[ Nu = \frac{-\kappa \partial T}{\partial y} |_{\eta=0} = x - \sqrt{Re \bar{x} \theta'(0)} \quad \ldots \quad (18) \]

The Eq. (13) has an analytical solution\(^{7,8}\)
\[ f = \left(1 - e^{-\frac{\eta}{m}}\right) \quad \text{for} \quad 0 \leq k_1 < 1 \quad \ldots \quad (19) \]
where $m = \sqrt{\frac{1 - l}{1 - k_1}}$

The solution for Eq. (14) is
\[ \theta = \frac{M(q_0, 1 + s, \xi)}{M(q_0, 1 + s, -r)} \left( \frac{\xi}{r} \right)^p + \frac{1 + 2 l r q}{(4 - 2 r + \alpha r)} \left\{ \frac{M(q_0, 1 + s, \xi)}{M(q_0, 1 + s, -r)} \left( \frac{\xi}{r} \right)^p \left( \frac{\xi}{r} \right)^{-\frac{q}{r \cdot (1 - r)}} \right\} \quad \ldots \quad (20) \]
where $\xi = -re^{-mn}, \ z = Ec \bar{x}^2, p = -\frac{(r + s)}{2}$,
\[ q_0 = p - \beta, q_1 = p - 2, r = \frac{Pr}{m^2}, s = \sqrt{(r - 4 \alpha)}, \]
\[ M(a, b, z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \quad \text{is the Kummer's function which satisfies the differential equation}, \quad \begin{align*}
\frac{dz^2}{dz^2} + \frac{(b - z)}{z} \frac{dM}{dz} - aM &= 0; \\
\end{align*} \]
and the Pochhammer symbol, $(a)_n = a(a+1) \ldots (a+n-1)$.

For large $z$,
\[ M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \left[ 1 + O \left( \frac{1}{z} \right) \right] \]

The nondimensional surface velocity and temperature gradients obtained from Eqs (19) and (20) are
\[ f''(0) = -m \quad \ldots \quad (21) \]
\[ \theta'(0) = -r m \left( \frac{P}{r} - \left\{ \frac{q_0}{1+s} \frac{M(q_0 + 1,2 + s, -r)}{M(q_0 + 1,2 + s, -r)} \right\} \frac{(1+2I)r \zeta}{1+s} \right) \]

When \( r \to 4/(2-\alpha) \), \( \theta \) and \( \theta(0) \) in Eqs (20) and (22) tend to

\[ \theta \to \left( \frac{\xi}{r} \right)^2 \left( \frac{M(2-\beta,1+s,\xi)}{M(2-\beta,1+s,-r)} \right) \frac{(1+2I)r \zeta}{s} \left\{ \ln \left( \frac{\xi}{r} \right) + \frac{1}{(1+s)} \int_{0}^{1} M(1,2+s,t) \, dt \right\} \] ... (23)

\[ \theta'(0) = -r m \left( \frac{2-\beta}{r} \right) \frac{M(3-\beta,2+s,-r)}{M(2-\beta,1+s,-r)} \frac{(1+2I)r \zeta}{s} \left\{ 1 - \left( \frac{r}{1+s} \right) M(1,2+s, -r) \right\} \] ... (24)

The asymptotic results for the temperature function \( \theta(\eta) \) for large Prandtl numbers are obtained as follows

Defining

\[ \theta = \left( \frac{(1+2I)r \zeta}{2-\beta r - \alpha} \right) e^{-2m \eta} + \Theta, \] ... (25)

the boundary layer Eq. (14) for energy and the corresponding boundary conditions in Eqs (15) and (16) are transferred to

\[ \frac{1}{Pr} \Theta'' + f \Theta' + (\alpha - \beta \gamma f') \Theta = (2-\beta \gamma) \zeta f' \frac{\partial \Theta}{\partial \zeta} \] ... (26)

\[ \Theta = 1 - \left( \frac{1+2I}{2-\beta r - \alpha} \right) \zeta \] at \( \eta = 0 \) ... (27)

\[ \Theta \to 0 \text{ as } \eta \to \infty \] ... (28)

Since the thermal boundary layer thickness is of the order \( \frac{1}{PrRe} \), the transformation co-ordinate \( \eta \) in Eq. (26) is further modified to

\[ \tau = \sqrt{Pr} \eta \] ... (29)

Using the Eq. (29), the stream function \( f(\eta) \) as well as its derivative, and Eqs (26)-(28) can be written for large Prandtl number as

\[ \sqrt{Pr} f(\eta) = \tau \] ... (30)

\[ f'(\eta) = 1 \] ... (31)

\[ \Theta + r \Theta + (\alpha - \beta \gamma) \Theta = (2-\beta \gamma) \zeta \frac{\partial \Theta}{\partial \zeta} \] ... (32)

\[ \Theta = 1 - \left( \frac{1+2I}{2-\beta r - \alpha} \right) \zeta \] at \( \tau = 0 \) ... (33)

\[ \Theta \to 0 \text{ as } \tau \to \infty \] ... (34)

Here dots denote differentiation with respect to \( \tau \).

The solution for Eqs (32)-(34) is

\[ \Theta = e^{-\frac{\tau^2}{r}} \left[ \Gamma(a) \beta \right] \frac{\Gamma(a)}{\Gamma(a)} \left( \frac{1+2I}{2-\beta r - \alpha} \right) \zeta \left( \frac{1+2I}{2-\beta r - \alpha} \right) \tau \frac{\Gamma(a) \beta}{\Gamma(a)} \] ... (35)

where \( a_1 = \frac{(\beta - \alpha + 1)}{2}, a_2 = \frac{(\beta - \alpha + 2)}{2}, \) \( a_3 = \frac{(3-\alpha)}{2}, \) and \( a_4 = \frac{(4-\alpha)}{2}. \)

The nondimensional surface temperature gradient obtained from Eqs (25) and (35) is

\[ \theta'(0) = \frac{-2(1+2I)m \zeta}{2-\beta r - \alpha} \sqrt{2Pr} \left[ \frac{\Gamma(a)}{\Gamma(a)} \right] \frac{(1+2I)\zeta}{(2-4 r - \alpha)} \] ... (36)

**Conclusion**—The important physical quantities such as skin-friction co-efficient and the heat transfer co-efficient can be found from Eqs (17) and (18) by substituting the nondimensional surface velocity gradient \( f'(0) \) and the surface temperature gradient \( \theta'(0) \) for the specified values of the non-Newtonian parameter \( k_1 \), Prandtl number \( Pr \), the local frictional heating parameter or Eckert number \( \zeta \), the heat source or sink parameter \( \alpha \), the constant power \( \beta \) which is related to the surface temperature, and the magnetic interaction parameter \( l \). These analytical solutions can provide not only a check against the finite difference/finite element model, but also a means by which the effect of a parameter change can be readily gauged, which is useful in understanding the flow phenomena.
References