Thermal stress analysis of a solid rocket motor nozzle throat insert using finite element method

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A finite element formulation is developed for the analysis of axisymmetric, transient, anisotropic heat conduction problem with the temperature dependant thermo-physical material properties of a graphite throat nozzle for the solid rocket motor. A standard Galerkin method using linear triangular element is employed for the space discretization. The time integration is done using an implicit time marching scheme of the first order differential equation. The convective heat transfer coefficient is calculated using the Bartz correlation. A thermal stress analysis is also carried out on the graphite throat of the nozzle using finite element method with two degrees of freedom. The developed computer codes for this purpose are validated with known analytical solution and available ANSYS code.

A convergent-divergent nozzle is used as a propulsive device in the launch vehicle. The nozzle expands high pressure and temperature gases from subsonic to supersonic velocities. Modern high energy solid propellant produces combustion gases of high temperature in order to enhance the specific impulse. The throat region of the convergent-divergent nozzle is therefore, exposed to high temperature and pressure environment as compared to other zone of the nozzle. The structural failure of the nozzle material may occur if the temperature exceeds the permissible operating design limit. A thermo-structural analysis is required for satisfactory performance of the solid rocket motor.

Analytical and numerical methods are available to obtain temperature distribution inside the material. Carslaw and Jaeger have obtained solution for a simple geometrical configuration. A numerical algorithm to solve heat conduction problem employing finite difference method is described in detail by Rosenberg. The finite difference approach needs jacobian transformation to change discretization from physical to computational domain, or special mathematical treatment to be done to take into an account irregular boundary of the nozzle wall.

Swaminathan and Rajagopalan have analysed the nozzle heat transfer problem using finite difference method. Lee has obtained temperature distribution inside a nozzle of a solid rocket motor employing finite difference method. Henderson has computed in-depth temperature response by solving heat conduction equation with temperature-dependent thermal properties.

The finite element method can easily solve a heat conduction problem for a complex geometrical configuration. Various grid arrangement can be used to accommodate a complex geometrical configuration and resulting simultaneous equations can be solved by using standard algorithm. The finite element technique to solve a heat transfer problem is reported elsewhere in detail. A two-dimensional heat conduction equation with time dependent heating condition at one surface of the specimen and a radiation boundary condition at the other end is solved using finite element method by Mehta et al.

The stress analysis of axisymmetric configuration is described by Zienkiewicz. To the authors' best knowledge, a coupled thermo-structure analysis of a rocket nozzle is not available in open literature. The main aim of the present paper is to investigate thermo-structural analysis of a graphite throat insert of a typical solid rocket motor nozzle. A finite element analysis is carried out to solve axisymmetric transient anisotropic heat conduction equation with temperature dependant thermal properties.
Temperature distributions are computed at each node point and at each time interval for a given convective heat transfer coefficient variation. Using the computed temperature profile at the end of motor burn-out time, the displacements and resulting thermal stresses are computed using finite element method. This coupled thermo structural analysis is validated using ANSYS software.

Heat Transfer Analysis

The governing axisymmetric transient anisotropic heat conduction equation with temperature dependant thermal properties can be written as

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r k_r \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left( k_z \frac{\partial T}{\partial z} \right) = \frac{\rho C_p}{\partial t} \frac{\partial T}{\partial t} \quad \ldots (1)$$

with the following initial and boundary conditions:

$$T = T_0 \quad \text{on region } i \quad \ldots (2)$$

and

$$-k \frac{\partial T}{\partial n} = h(T_{\text{ave}} - T) \quad \text{on surface } \Gamma \quad \ldots (3)$$

where $k_r$ and $k_z$ are the thermal conductivity in radial and axial directions, respectively. $\rho$ is the density of the material, $C_p$ is the specific heat of the material, $t$ is time, $h$ is convective heat transfer, $n$ is the outward normal and $T_{\text{ave}}$ is combustion gas temperature. The other sides of the nozzle wall are insulated.

The spacewise discretization of Eq. (1) subjected to the above boundary condition can be accomplished using Galerkin method. Let the unknown function temperature $T$ be approximated throughout the solution domain at any time $t$ by the relationship

$$T = \sum_{i=1}^{n} N_i(r,z)T_i(t) \quad \ldots (4)$$

where $N_i$ is usual shape function defined element by element, $T_i$ being the nodal parameters. For a typical triangular element $e$ with nodes numbered anticlockwise as $i, j, k$ and placed at vertices of the triangle, the shape function $N_i$ is

$$N_i = a_i + b_i r + c_i z \quad \ldots (5)$$

where

$$a_i = \frac{(r_j z_k - r_k z_j)}{2 A^e} \quad \ldots (6)$$

$$b_i = \frac{(z_j - z_k)}{2 A^e} \quad \ldots (6)$$

$$c_i = \frac{(r_k - r_j)}{2 A^e} \quad \ldots (6)$$

and $A^e$ is the area of the element, $e$.

The general procedure for solving Eq. (1) is to evaluate the Galerkin residual integral with respect to space coordinate for a fixed instant of time. This yields a system of ordinary differential equation solved to obtain temperature distribution. The $n$ equations can be written in a matrix form as

$$[C] \frac{dT}{dt} + [K]T = \{f\} \quad \ldots (7)$$

where $[C]$, $[K]$ and $\{f\}$ are capacitance, stiffness and load vector matrix, and can be written as

$$[C] = \int_V \rho C_p [N]^T [N] dV \quad \ldots (8)$$

$$[K] = \int_V \left[ k_r \frac{\partial [N]^T}{\partial r} \frac{\partial [N]}{\partial r} + k_z \frac{\partial [N]^T}{\partial z} \frac{\partial [N]}{\partial z} \right] dV \quad \ldots (9)$$

$$f = \int_V \left[ k_r \frac{\partial [N]^T}{\partial r} \frac{\partial T}{\partial r} + k_z \frac{\partial [N]^T}{\partial z} \frac{\partial T}{\partial z} \right] dV \quad \ldots (10)$$

where $V$ is the volume of the element that is equal to $2\pi r_c A^e$ about z axis, $r_c$ is the centroid of the triangular element.

Time integration

The set of ordinary differential Eq. (7) which defines the discretized problem is solved using implicit time marching a scheme. The equation can be written for time level $n$ to $n+1$ as

$$[C] + [P]T^{n+1} = [C]T^n + \frac{f^n}{\Delta t} \quad \ldots (11)$$

where $[P]$ is combination of $[C]$ and $[K]$ matrix. The algorithm is unconditionally stable and convergent in the context of finite difference. Numerical experiments have been done at different values of aspect ratio and time-step in order to verify the stability of the algorithm.

Grid generation

The finite element solver requires the element coordinate and connectivity in the domain of computation. An algebraic procedure is used to generate a triangular element. A schematic sketch of the nozzle insert is displayed in Fig.1. The outer wall and inner wall of the nozzle insert are represented as $r_{\text{top}}(z)$ and $r_{\text{w}}(z)$. The radial increment is computed as difference of inner and outer wall radius divided by number of sub division. The grid is structured. Therefore, the connectivity of the triangular elements is obtained using recurrence relation. Fig. 2 shows typical grid arrangement in the nozzle insert.

Numerical stress analysis

In the axisymmetric problem of a nozzle throat insert, a cylindrical coordinate system is taken in
computation of thermal stress. The strain equation for the axisymmetric configuration can be written as

\[(\epsilon')' = [\epsilon_r \; \epsilon_\theta \; \epsilon_z \; \epsilon_r] \quad \ldots \quad (12)\]

\[(\epsilon')' = [\epsilon_r \; \epsilon_\theta \; \epsilon_z \; \epsilon_r] \quad \ldots \quad (13)\]

\[(\epsilon')' = [\alpha \Delta T \; \alpha \Delta T \; \alpha \Delta T] \quad \ldots \quad (14)\]

where \(\{\epsilon\}\) is the total strain, \(\{\epsilon\}\) is the elastic strain and \(\{\epsilon\}\) is the thermal strain, \(\alpha\) is the coefficient of thermal expansion and \(\Delta T\) is the change of temperature. The above elastic and thermal strain vector can be written as

\[\{\epsilon\} = \{\epsilon\} + \{\epsilon\} \quad \ldots \quad (15)\]

the stress can be calculated using following Hooke's law:

\[\{\sigma\} = [D] \{\epsilon\} \quad \ldots \quad (16)\]

where \([D]\) is the material stiffness matrix and can be written as

\[\{u\} = f(r, z), \quad \{w\} = g(r, z) \quad \ldots \quad (19)\]

where \(u, v, w\) and \(u, v, w\) are displacement in \(r, \theta, z\) directions, respectively. The strain displacement relationship is given as

\[e_r = \frac{\partial u}{\partial r}, \quad e_\theta = \frac{u}{r}, \quad e_z = \frac{\partial w}{\partial z} \quad \ldots \quad (20)\]

\[e_r = 0, \quad e_\theta = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad e_z = 0\]

The unknown displacements can be written in terms of element nodal values

\[\begin{bmatrix} u(r, z) \\ u(z, r) \\ w(r, z) \end{bmatrix} = \begin{bmatrix} N_i \; N_j \; 0 \; N_k \; 0 \end{bmatrix} \begin{bmatrix} u_{i} \\ u_{j} \\ u_{k} \end{bmatrix} \quad \ldots \quad (21)\]
It can be written in matrix form as
\[ \{u\} = [N]\{u^e\} \quad \ldots \quad (22) \]

By differentiating equation 21 using the strain displacement relationship, we get
\[ (e) = \frac{I}{2A} \begin{bmatrix} b_1 & 0 & b_j & 0 & b_k & 0 \\ 0 & \frac{2A N_j}{r} & 0 & \frac{2A N_j}{r} & 0 & \frac{2A N_k}{r} \\ 0 & 0 & c_i & 0 & c_i & 0 \\ c_i & b_j & c_i & b_j & c_k & b_k \end{bmatrix} \{u^e\} \quad \ldots \quad (23) \]

It can be written in matrix form as
\[ \{e\} = [B]\{u^e\} \quad \ldots \quad (24) \]

The element stiffness matrix can be given as
\[ [K^e] = [B]^T [D] [B] \, dv \quad \ldots \quad (25) \]

\[ [K^e] = \frac{2\pi A \, r_c \, [B]^T [D] [B]} \quad \ldots \quad (26) \]

The column vector associated with the thermal change is
\[ J_v [B] [D] \{\varepsilon_T\} \, dv = 2\pi A \, r_c [B]^T [D] \{\varepsilon_T\} \quad \ldots \quad (27) \]

The integral involving the boundary condition for a load vector can be expressed as
\[ f^p_r = \int_T [N] \begin{bmatrix} p_r \\ p_z \end{bmatrix} \, d\Gamma \quad \ldots \quad (28) \]

where \( p_r \) and \( p_z \) are the surface stresses in the \( r \) and \( z \) directions.

The element stress components are calculated after finding out the nodal displacements by finite element method, then by Hooke’s law
\[ \{\sigma\} = D(\{e\} - \{\varepsilon_T\}) \quad \ldots \quad (29) \]

The stress components as a function of the element a nodal displacements and thermal strain vector are
\[ \{\sigma^e\} = D([B]\{u^e\} - \{\varepsilon_T\}) \quad \ldots \quad (30) \]

**Results and Discussion**

Using the above finite element formulation, the thermostructural problem of a graphite throat nozzle is analysed. The computer programme developed is validated for heat transfer and thermal stresses separately. The validation cases are described in detail elsewhere \(^{10}\).

Table 1 gives the geometrical parameters of G1 and G2 for typical nozzle throat inserts of a solid rocket motor. The nozzle G1 was modelled by 456 nodes and 784 triangular elements while G2 was modelled by 648 nodes and 1166 triangular elements. The non-linear temperature dependent thermo-physical and other material properties of graphite were taken from Table 2. The convective heat transfer coefficient for the analysis is calculated from Bartz correlation\(^{11}\)

\[ h = \frac{0.026 \mu a C_{p}}{D_{m}^{0.2} \, Pr^{0.6}} \left( \frac{P_c}{\bar{C}} \right)^{0.8} \left( \frac{D_{m}}{R_{e}} \right)^{0.4} \left( \frac{A_{m}}{A_{e}^{0.9}} \right)^{0.15} \sigma \quad \ldots \quad (31) \]

and

\[ \sigma = \frac{T_{e}}{2T_{c}} \left[ 1 + \frac{\gamma - 1}{2} M^2 \right]^{0.15} \ldots \quad (32) \]

where \( M \) is Mach number and \( T_{e} \) local gas temperature. The characteristic velocity \( C' \), the

<table>
<thead>
<tr>
<th>Table 1—Geometrical parameters of nozzle throat inserts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type and Identification of Nozzle</td>
</tr>
<tr>
<td>Graphite Throat, G1</td>
</tr>
<tr>
<td>Graphite Throat, G2</td>
</tr>
</tbody>
</table>

Table 2—Properties of graphite\(^{12}\)

<table>
<thead>
<tr>
<th>Property</th>
<th>Graphite</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal conductivity</td>
<td>( X_0 ) + 7.07390E+01</td>
</tr>
<tr>
<td>( K_a(T) )</td>
<td>( X_1 ) - 3.76325E-02</td>
</tr>
<tr>
<td>( K_b(T) )</td>
<td>( X_2 ) + 9.58421E-06</td>
</tr>
<tr>
<td>( K_c(T) )</td>
<td>( X_3 ) - 8.18337E-10</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>( Y_0 ) - 7.10309E+01</td>
</tr>
<tr>
<td>( K_a(T) )</td>
<td>( Y_1 ) - 3.60873E-02</td>
</tr>
<tr>
<td>( K_b(T) )</td>
<td>( Y_2 ) + 7.14803E-06</td>
</tr>
<tr>
<td>( K_c(T) )</td>
<td>( Y_3 ) + 1.69792E-10</td>
</tr>
<tr>
<td>Density</td>
<td>( Z_0 ) - 5.991012E+02</td>
</tr>
<tr>
<td>Coefficient of thermal expansion</td>
<td>( Z_1 ) - 4.76525E+00</td>
</tr>
<tr>
<td>Modulus of Elasticity</td>
<td>( Z_2 ) - 3.08595E-03</td>
</tr>
<tr>
<td>Specific heat</td>
<td>( Z_3 ) + 6.56786E-07</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>1700.0</td>
</tr>
<tr>
<td>Coefficient of thermal expansion</td>
<td>4.87X10^-6</td>
</tr>
<tr>
<td>Modulus of Elasticity</td>
<td>9.20X10^9</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>0.11</td>
</tr>
</tbody>
</table>

\[ K_a(T) = X_0 + X_1 T + X_2 T^2 + X_3 T^3 \, W/mK \]

\[ K_a(T) = Y_0 + Y_1 T + Y_2 T^2 + Y_3 T^3 \, W/mK \]

\[ C_a(T) = Z_0 + Z_1 T + Z_2 T^2 + Z_3 T^3 \, J/kg K \]
Table 3—Properties of combustion products

<table>
<thead>
<tr>
<th>Properties</th>
<th>HTPB+NH₄ClO₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chamber Pressure, ( P_c ) (N/m²)</td>
<td>4.415E+06</td>
</tr>
<tr>
<td>Chamber Temperature, ( T_c )(K)</td>
<td>3410.0</td>
</tr>
<tr>
<td>Specific heat ( C_p )(J/kgK)</td>
<td>1716.55</td>
</tr>
<tr>
<td>Ratio of specific heats ( \gamma )</td>
<td>1.15</td>
</tr>
<tr>
<td>Characteristic velocity ( C^* )(m/s)</td>
<td>1582.25</td>
</tr>
<tr>
<td>Molecular weight, ( W )(kg/kmol)</td>
<td>29.0</td>
</tr>
</tbody>
</table>

Specific heat, \( C_p \) and chamber temperature \( T_c \) are computed from combustion gases in frozen composition. Chamber pressure \( P_c \) corresponds to the test condition and the combustion gas properties were used for the convective heat flux calculation are taken from Table 3.

The temperature distribution on the nozzle wall of graphite throat nozzle G1 is shown in Fig. 3 and the isothermal contour plot at various locations are shown in Fig. 4. The throat region of the nozzle is attained a maximum temperature of 2977 K after the specified period of performance of the motor. The convergent region experiences a temperature of the range of 2825-2950 K and a divergent region is having a range of 2925-2700 K. The convective heating of the nozzle liner causes, the high temperature at the inner wall and the maximum value of convective heat transfer coefficient is at the throat region. The minimum temperature is at the outer wall and at the region corresponding to throat and nearby. The maximum throat material at this region is the reason behind this minimum temperature. In a divergent region, this temperature increases due to less throat material. The maximum heat transfer coefficient of 14900 W/m K is at the throat region and minimum at the exit region of the divergent of the nozzle which are calculated employing the above mentioned Bartz’s correlation. The heat transfer coefficient increases from the convergent region up to throat and then decreases.

In nozzle G2, the temperature distribution of the walls of a nozzle and isothermal contour plots are as in Figs 5 and 6. Here also, the maximum temperature of 3014 K, which is more than the first case, is at the throat region. The minimum temperature of 303 K is at the outer wall region corresponding to the throat area where maximum throat material exists. The convective heat transfer coefficient plot is similar like G1, nozzle. By comparing nozzle G1 and G2, it reveals that the heat transfer behaviour of nozzles are similar.

The model taken for the structural analysis is the
G1 nozzle. The temperature distribution at all nodal points determined by thermal analysis is an input for the structural analysis purpose along with the pressure applied inside the boundary of the nozzle as shown in Fig. 7. The zero displacement on the lateral boundary is considered assuming a rigid assembly of the nozzle insert with the back-up material. The pressure distribution is assumed constant along the inner wall of the nozzle. However, the calculation can be carried out with the varying pressure distribution. The boundary condition for structural analysis is also shown in the same figure.

The nodal displacement at each node is determined in $z$ and $r$ directions. Displacement $v$ is taken zero due to the axisymmetry condition. Maximum value of displacement is 0.040697 mm noticed at the node, which is the last corner node of the inner surface of the nozzle. At this node region, the boundary is free to expand. The radial direction displacement is maximum at the nodal point near the throat region and value is 0.12236 mm.

The distribution of stresses are obtained such as $\sigma_z$, $\sigma_r$, and $\sigma_0$. For the sake of the brevity, we are presenting the $\sigma_r$ and $\sigma_z$ in Figs 8 and 9. In all the these cases, the compressive stresses are acting on the inner surface of the nozzle and change to tensile towards the outer wall. It is noticed that the throat region and the inner surface are subjected to high compressive stress. The inner surface is under the compressive stresses and changes to tensile stresses toward the outer surface.

The stress values obtained by the finite element
method and the computer code is validated by a software ANSYS 4.4, which deals with the thermal and structural problems. The model selected for the validation is same as that of nozzle GI. The boundary conditions applied are also the same. The stress plots are shown in Fig. 10. Comparing the results obtained by the computer code and ANSYS program, it is understood that they are in good agreement with each other.

Conclusions
In this thermostructural analysis of a rocket nozzle, a numerical method has been developed for solving an axisymmetric, transient, anisotropic heat conduction equation with temperature dependent thermo-physical properties. For the numerical simulation finite element method is used due to its ability to handle the arbitrary geometry, initial conditions, boundary conditions and material properties. Linear triangular elements have been used for the space discretization. An implicit time marching scheme is used for time integration of the system of first order differential equation. The displacements and stresses formed inside the graphite throat nozzle is calculated using finite element method in conjunction with temperature distribution obtained from the thermal analysis. The computer program generated for the solution of thermal analysis is validated against an analytical solution available. The numerical values are found in good agreement with the analytical solutions. An ANSYS 4.4 software is used to validate the computer program for stress calculation. The computed results are found in good agreement with the software results.

References
9 ANSYS 4.4, Software, developed by Swanson Analysis Corporation, USA.  