Information and complexity measures for the ring-shaped modified Kratzer potential†

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Received 30 April 2014; revised and accepted 9 May 2014

In this study, the Fisher information measure, Shannon entropy, Renyi entropy, Tsallis entropy and Fisher-Shannon complexity of the ring-shaped modified Kratzer potential are investigated. The trends in the variation of the information and complexity measures considered for this model quantum system are discussed.

Keywords: Theoretical chemistry, Fisher information, Shannon entropy, Tsallis entropy, Renyi entropy, Fisher-Shannon complexity, Ring-shaped modified Kratzer potential

There has been an interest in studies on information-theoretic and complexity measures for quantum model systems in the recent years1-10. The information entropy for the trigonometric Rosen-Morse potential2, harmonic oscillator11-14, hydrogenic atoms15, Morse potential16,17, Yukawa and Hulthen potentials18 have been investigated.

The Kratzer potential plays an important role in atomic and molecular physics and quantum chemistry19-22. Cheng and Dai23-25 proposed a new potential consisting of the modified Kratzer potential plus a new proposed ring-shaped potential which has possible applications to ring-shaped organic molecules like cyclic polyenes and benzene23-25. In this work, we shall obtain the Fisher information, Fisher length, Shannon entropy, Shannon length, Renyi entropy, Renyi length, Tsallis entropy and the Fisher-Shannon complexity for the ring-shaped modified Kratzer potential. In this section, we present the definition of the information theoretical measures used in this work which are based on the 1-normalized electron density in the position and momentum space. The Shannon information entropy in the position space is given by Eq. (1)11,14,26,27.

\[ S(\rho) = - \int \rho(\mathbf{r}) \log \rho(\mathbf{r}) d\mathbf{r} \] … (1)

The Shannon entropy measures the uncertainty in the localization of a particle in space. The lower the entropy, the more concentrated is the wave function. The smaller the uncertainty, the higher is the accuracy in predicting the localization of the particle. For applications of the Shannon entropy, see Ref. 26. The Shannon length (SL) is defined as Eq. (2)1,28,29.

\[ \frac{4\pi}{3} S_L^3 = e^{S(\rho)} \] … (2)

The Fisher information measure is defined by Eq. (3)27.

\[ I(\rho) = \int \frac{[\nabla \rho(\mathbf{r})]^2}{\rho(\mathbf{r})} d\mathbf{r} \] … (3)

The Fisher information measure is used as a tool for characterizing complex signals or systems with applications in geophysics, biology, atomic physics, quantum chemistry and other related areas1,4-10,30-41. The Fisher length (IL) is defined as Eq. (4)1,28,29.

\[ I_L = (I(\rho))^{1/2} \] … (4)

The Tsallis entropy is defined as Eq. (5),

\[ T_q[\rho] = \frac{1}{q-1} \left( 1 - \int \rho(x)^q \, dx \right), \quad q > 0, \quad q \neq 1 \] … (5)
and describes non-extensive systems. It has been applied to tissue radiation response, solar neutrino problem and image thresholding. The Renyi entropy defined as Eq. (6),

\[ R_q[\rho] = \frac{1}{1-q} \ln \left( \int \rho(x)^q \, dx \right), \quad 0 < q < \infty, \quad q \neq 1 \quad \ldots \quad (6) \]

is a generalization of the Shannon entropy. The Renyi length \( L_r \) is defined as Eq. (7)\(^{1,28,59}\).

\[ \frac{4\pi}{3} R_e^3 = e^{R[\rho]} \quad \ldots \quad (7) \]

The Fisher-Shannon complexity defined as Eq. (8)\(^{50-52}\).

\[ FS = I[\rho] / I[\rho] \quad \ldots \quad (8) \]

is a statistical complexity measure which is built upon a global information measure combined with a local information measure. Here, \( I[\rho] \) is the Fisher information measure, and

\[ J[\rho] = \frac{1}{2\pi e} \quad \ldots \quad (9) \]

where \( S[\rho] \) is the Shannon information entropy. The Fisher-Shannon measure grasps the oscillatory nature of the electronic cloud together with its total extent in the configuration space\(^{53}\). Other forms of complexity measures are the algorithmic complexity\(^{50,54,55}\), the statistical measure of complexity \( C \), defined by Lopez-Ruiz, Mancini, Calbet (LMC)\(^{50,56-59}\) and the two parameter disorder-order derived measure of complexity \( \Gamma_{a,b} \) according to Shiner, Davison, Landsberg (SDL)\(^{50,60,61}\). Both \( C \) and \( \Gamma_{a,b} \) are products of two global information measures\(^{50}\). They quantify different facets of the internal disorder of a system which are manifest in the diverse and complex three-dimensional geometries of its orbitals\(^{53}\).

In what follows, the solutions of the Klein-Gordon equation with the ring-shaped modified Kratzer potential are given. This is followed by the Fisher information measure and Shannon entropy for the ring-shaped modified Kratzer potential. The Tsallis and Renyi entropies for the ring-shaped modified Kratzer potential are then discussed followed by the Fisher-Shannon complexity and concluding remarks.

**Solution of the Schrödinger Equation with the Ring-Shaped Modified Kratzer Potential**

The solution of the Schrödinger equation with the Ring-Shaped Modified Kratzer potential has been obtained in Ref. 62. We shall review these solutions briefly in this section to define our notations. The Schrödinger equation with equal scalar and vector potentials can be written as Eq. (10)\(^{62}\).

\[ \left[ -\frac{\hbar^2}{2m} \nabla^2 \right] \Psi(r, \theta, \phi) = \left[ E - V(r, \theta) \right] \Psi(r, \theta, \phi) \quad \ldots \quad (10) \]

The potential \( V(r, \theta) \), in this case, is the ring-shaped modified Kratzer potential which is given as Eq. (11)\(^{23,24,62}\).

\[ V(r, \theta) = D_e \left( \frac{r - r_c}{r} \right)^2 + \beta^2 \cos^2 \theta \quad r^2 \sin^2 \theta \quad \ldots \quad (11) \]

where \( \beta \) is a positive real constant, \( D_e \) is the dissociation energy and \( r_c \) is the equilibrium internuclear separation.

Taking the wave functions to be of the form,

\[ \Psi(r, \theta, \phi) = \frac{u(r)}{r} H_n(\theta)e^{i\phi}, \quad m = 0, \pm 1, \pm 2, \ldots \quad \ldots \quad (12) \]

\[ H_n(x) = N_n \left( 1 - x^2 \right)^{-\alpha/2} \frac{d}{dx} \left[ \frac{d^{l+m}}{dx^{l+m}} \left( x^2 - 1 \right)^l \left( 1 - x^2 \right)^m \right] \quad \ldots \quad (13) \]

where \( H_n(x) \) stands for the associated-Legendre functions \( P_{l+m}^m(x) \) and \( N_n \) is the normalization constant. The wave function is obtained as Eq. (13).

\[ \Psi(r, \theta, \phi) = C_{NL}(2\epsilon r)^{\alpha/2} e^{-\epsilon r} L_{2l+1}^{2l+1}(2\epsilon r) Y_{l+m}(\Omega) \quad \ldots \quad (14) \]

where

\[ C_{NL} = \sqrt{\frac{(2\epsilon)^3 N!}{2(N+L+1)\Gamma(N+2L+2)}}, \quad N = 0, 1, 2, \ldots \quad \ldots \quad (15) \]

\[ L = \sqrt{(2l'+1)^2 + 8\mu(D_e(r_c)^2 - \beta)} - 1 \quad \ldots \quad (16) \]

\[ \epsilon = \frac{2\mu D_e r_c}{N} \quad \ldots \quad (17) \]

\[ N' = \left[ 2N + 1 + \sqrt{(2l'+1)^2 + 8\mu(D_e(r_c)^2 - \beta)} \right] \quad \ldots \quad (18) \]

\( N \) is the principal quantum number and \( Y_{l+m}(\Omega) \) is the spherical harmonics given as Eq. (19).

\[ Y_{l+m}(\Omega) = Y_{l+m}(\theta, \phi) = N_{l+m}P_{l+m}^m(\cos \theta)e^{i\phi} \quad \ldots \quad (19) \]

and the normalization constant \( N_{l+m} \) is obtained as Eq. (20).
\[ N_{\ell m} = \sqrt{\frac{(2\ell' + 1)(\ell' + m')!}{4\pi (\ell' - m')!}} \quad \ldots (20) \]

**The Fisher and Shannon Information Measures for the Ring-Shaped Modified Kratzer Potential**

The Fisher information measure for the ring-shaped modified Kratzer potential will be obtained by using Eq. (3). The probability density of this potential model is given as Eq. (21).

\[ \rho(r) = C N L^2 e^{-2\epsilon r} \left| \mathcal{L}_{2n+1}^2 (2\epsilon r) \right|^2 \left| \mathcal{Y}_{\ell m'}(\Omega) \right|^2 \quad \ldots (21) \]

Taking into consideration the gradient operator \( \nabla \) in polar coordinates and the azimuthal independence of the probability density of the ring-shaped modified Kratzer potential, one has the following expression for the Fisher information measure (Eq. (22),

\[ I(\rho) = \int \frac{1}{\rho(r)} \left[ \frac{\partial \rho(r)}{\partial r} \right]^2 \, dr + \int \frac{1}{\rho(r)} \left[ \frac{1}{r} \frac{\partial \rho(r)}{\partial \theta} \right]^2 \, dr = I_1 + I_2 \quad \ldots (22) \]

where

\[ I_1 = \int \frac{1}{\rho(r)} \left[ \frac{\partial \rho(r)}{\partial r} \right]^2 \, dr \quad \ldots (23) \]

and

\[ I_2 = \int \frac{1}{\rho(r)} \left[ \frac{1}{r} \frac{\partial \rho(r)}{\partial \theta} \right]^2 \, dr \quad \ldots (24) \]

The integral \( I_1 \) can be decomposed into a sum of five integrals denoted by \( I_{1i} \), \( i = 1 - 5 \). By considering the condition in Appendix (A13) and on using (A1), (A2) and (A13), one has the following integrals:

\[ I_{11} = (2\epsilon)^2 \frac{2L^2}{(N + L + 1)(2L + 1)} \quad \ldots (25) \]

\[ I_{12} = -(2\epsilon)^2 \frac{2L}{N + L + 1} \quad \ldots (26) \]

\[ I_{13} = (2\epsilon)^2 \quad \ldots (27) \]

\[ I_{14} = (2\epsilon)^2 \frac{2N}{N + L + 1} \quad \ldots (28) \]

\[ I_{15} = -(2\epsilon)^2 \frac{2N}{N + L + 1} \quad \ldots (29) \]

Therefore, putting together the computed values for \( I_{1i} \), we have

\[ I_1 = (2\epsilon)^2 \frac{2NL + L + N + 1}{(2L + 1)(N + L + 1)} \quad \ldots (30) \]

Also, the integral \( I_2 \) can be decomposed into the sum of three integrals denoted by \( I_{2i} \), \( i = 1 - 3 \). Using equations (A7), (A8) and (A9), we obtain

\[ I_{21} = (2\epsilon)^2 \frac{2(N + L + 1)(2L + 1)}{4\pi \left[ (\ell' - m')! \right]^2 (2\ell' + 1)^2} \times \left( \frac{m' - \ell'}{2\ell' + 1} \right) \quad \ldots (31) \]

\[ I_{22} = (\ell' + m')^2 \frac{(2\ell' + 1)(\ell' + m')!}{4\pi \left[ (\ell' - m')! \right]^2 (2\ell' + 1)^2} \times \left( \frac{m' - \ell'}{2\ell' + 1} \right) \quad \ldots (32) \]

and

\[ I_{23} = (\ell' + m')^2 \frac{(2\ell' + 1)(\ell' + m')!}{2(N + L + 1)(2L + 1)} \times \left( \frac{m' - \ell'}{2\ell' + 1} \right) \quad \ldots (33) \]

Then, the integral \( I_2 \) is obtained as

\[ I_2 = (2\epsilon)^2 \frac{2(N + L + 1)(2L + 1)}{4\pi \left[ (\ell' - m')! \right]^2 (2\ell' + 1)^2} \times \left( \frac{\ell'^2}{m'} - \frac{2\ell'^2}{m'} + \frac{(\ell' + m')(\ell' - m')}{m'} - \frac{2\ell'(\ell' - m')}{m'} \right) \quad \ldots (34) \]

Hence, the following expression for the Fisher information measure is obtained (Eq. (35)).

\[ I(\rho) = I_1 + I_2 = (2\epsilon)^2 \frac{2NL + L + N + 1}{(2L + 1)(N + L + 1)} + (2\epsilon)^2 \frac{2(N + L + 1)(2L + 1)}{4\pi \left[ (\ell' - m')! \right]^2 (2\ell' + 1)^2} \times \left( \frac{\ell'^2}{m'} - \frac{2\ell'^2}{m'} + \frac{(\ell' + m')(\ell' - m')}{m'} - \frac{2\ell'(\ell' - m')}{m'} \right) \quad \ldots (35) \]
The graphs of the Fisher length against \(n\), \(D_e\) and \(r_e\) are plotted in Figs 1, 2 and 3, respectively, for \(\ell' = 0\). In Fig. 1, the Fisher length increases as \(n\) increases. It can be observed from Fig. 2 that the Fisher length decreases with the dissociation energy \(D_e\), while the Fisher length increases with increasing \(r_e\) in Fig. 3 except for the initial decrease from \(r_e = 1\) Å to \(r_e = 2\) Å.

The Shannon entropy of the ring-shaped modified Kratzer potential

The Shannon entropy of a hydrogen system in an arbitrary state characterized by the quantum numbers \((n, \ell', m)\) has been obtained as Eq. (36)\(^{11,14,27}\)

\[
S(\rho) = S_{NL}(R) + S_{\psi}(Y) \tag{36}
\]

For the case of the Kratzer potential, the Shannon entropy for the radial part is

\[
S_{\psi}(R) = \int_0^\infty r^2 R_{NL}^2(r) \ln R_{NL}^2(r) dr \tag{37}
\]

Using Eqs (42) and (43) of Yanez et al.\(^{11}\) and Eq. (5.1) of Dehesa et al.\(^{14}\), we obtain Eq. (38)

\[
S_{\psi}(R) = \frac{N!}{2(N + L + 1)(N + 2L + 1)!} \times \left[ \frac{(N + 2L + 3)!}{N!} + 4(N + 2L + 2)! + \frac{(N + 2L + 3)!}{(N - 2)!} \right]
- \ln \left[ \frac{(2\pi)^{\frac{3}{2}} N!}{2(N + L + 1)(N + 2L + 1)!} \right]
- 2L \left[ \frac{2N + 1}{2(N + L + 1)} + \psi(N + 2L + 2) \right]
+ \frac{N!}{2(N + L + 1)(N + 2L + 1)!} E_1(\ell_{NL}^2)
\tag{38}
\]

This can be simplified into the following form (Eq. (39)):

\[
S_{\psi}(R) = -3\ln e - \ln \left[ \frac{N!}{2(N + L + 1)(N + 2L + 1)!} \right]
+ 3(L + 1)^2 - 3N - 2L \left[ \frac{2N + 1}{2(N + L + 1)} + \psi(N + 2L + 2) \right]
+ \frac{N!}{2(N + L + 1)} E_1(\ell_{NL}^2)
\tag{39}
\]

where \(\psi(x) = \Gamma'/(x)\Gamma(x)\), and the symbol \(E_1(y_n)\) denotes the entropic integral\(^{15}\) (Eq. (40)).

\[
E_1(\ell_{NL}^2) = \int_0^\infty x\ell_{NL}^2(x)r^2 \ln r^2 dx \tag{40}
\]

Fig. 1 – Fisher length against \(n\) for \(\ell' = 0\), \(\beta = 1\), \(\mu = 0.5\), \(D_e = 6.780447246\) eV, \(r_e = 1.116\) Å.

Fig. 2 – Fisher length against \(D_e\) for \(\ell' = 0\), \(\beta = 1\), \(\mu = 0.5\), \(n = 2\), \(r_e = 1.116\) Å.

Fig. 3 – Fisher length against \(r_e\) for \(\ell' = 0\), \(\beta = 1\), \(\mu = 0.5\), \(n = 2\), \(D_e = 2\) eV.
To obtain $S_{c_m}(Y)$, we make use of the result (lower bound obtained for the Shannon entropy of the hyperspherical harmonics in 3D) in Eq. (32) of Ref. 63 and the fact that the Shannon entropy has the same value in the states $(n, l', m')$ and $(n, l' - m')^{11}$. Therefore, $S_{c_m}(Y)$ is obtained as Eq. (41),

$$S_{c_m}(Y) = -\ln \left[ \sum_{k=0}^{\infty} C_2(\ell', [m']) \left[ \left( -\ell' + [m'] + k, \left[ m' + k + 1/2, \left[ [m'] + k + 1, \left[ 2k + 3/2 \right] \right] \right] \right) \right] \right]$$ … (41)

where

$$C_2(\ell', [m']) = \frac{(4[m'] + 4k + 1)(2\ell' + 1)^2 \Gamma(k + 1/2) \Gamma(2[m'] + k + 1/2)}{16\pi^k (2[m'] + k)!} \times \left[ \frac{(\ell' + [m'] + k) \Gamma([m'] + k + 1/2)}{([m'] + k) \Gamma([m'] - k) \Gamma(2[m'] + 2k + 3/2)} \right]^2$$ … (42)

The graphs of the Shannon length against $n$, $D_e$ and $r_e$ are plotted in Figs 4-6 respectively, for $n = 0; \ell' = 0$. In Fig. 4, the Shannon length is initially relatively constant with increasing $n$, but there is an abrupt increase when $n = 15$. The reverse behaviour is observed when the Shannon length is plotted against the dissociation energy $D_e$. The Shannon length initially decreases with $D_e$ but is relatively constant from $D_e = 10$ eV upwards. The behaviour of the Shannon length against $r_e$ is similar to its behaviour against $n$ as can be observed from Figs 4 and 6.

The Tsallis and Renyi Entropies of the Ring-Shaped Modified Kratzer Potential

Using Eqs (5), (6) and (21), the Tsallis and Renyi entropies are obtained respectively as Eqs (43) and (44).

$$T_q[\rho] = \frac{1}{q-1} \left[ 1 - \int Y_{n,m}(\theta, \phi)^q \left( 2\varepsilon \right)^q \left( \frac{4\pi C_{nl}}{(2\varepsilon)^q} \right)^{1/2} \varepsilon^{2L+1} \varepsilon_N^{2L+1} (2\varepsilon r)^{2L+1} \varepsilon^{1/2} d\Omega \right] \right]$$ … (43)

$$T_q[\rho] = \frac{1}{q-1} \left[ 1 - \int Y_{n,m}(\theta, \phi)^q \left( 2\varepsilon \right)^q \left( \frac{4\pi C_{nl}}{(2\varepsilon)^q} \right)^{1/2} \varepsilon^{2L+1} \varepsilon_N^{2L+1} (2\varepsilon r)^{2L+1} \varepsilon^{1/2} d\Omega \right] \right]$$ … (44)
and
\[
R_q[\rho] = \frac{1}{1-q} \ln \left[ \frac{4\pi C_{N\ell}^2}{(2\epsilon)^2} \right] + \int_0^\infty \left[ I_{2q}^{(4)}(2\epsilon r) + |Y_{\ell m}(\theta,\phi)|^2 \right] d\Omega \]
\[\text{(44)}\]

By using the linearization formula of Srivastava-Niukkanen for the products of various Laguerre polynomials\textsuperscript{28,64,65}, we obtain Eq. (45)
\[
t^{a} L_n^{(a)}(x_1) \ldots L_n^{(a)}(x_n) = \sum_{k=0}^{n} \gamma_k(\mu; x_1, \ldots, x_n) L_n^{(a)}(t)
\]
\[\text{(45)}\]

where
\[
\gamma_k(\mu; x_1, \ldots, x_n) = (\alpha+1)_k \left( \frac{\alpha_1 + m_1}{m_1} \right) \ldots \left( \frac{\alpha_n + m_n}{m_n} \right)
F_{\alpha+1}^{(n+1)}(\alpha + \mu + 1, -m_1, \ldots, -m_n, -k; \alpha_1 + 1, \ldots, \alpha_n + 1, \alpha + 1; x_1, \ldots, x_n, 1)
\]
\[\text{(46)}\]

Hence, Tsallis and Renyi entropies are obtained in terms of one of Lauricella's hypergeometric functions of \((n + 1)\) variables (Eqs 47 & 48),
\[
T_q[\rho] = \frac{1}{q-1} \left[ 1 - \frac{4\pi C_{N\ell}^2}{(2\epsilon)^2} \right] \sum_{k=0}^{n} \gamma_k(4Lq; \frac{1}{q}, \ldots, \frac{1}{q}) L_n^{(a)}(t)
\]
\[\text{(47)}\]

and
\[
R_q[\rho] = \frac{1}{1-q} \ln \left[ \frac{4\pi C_{N\ell}^2}{(2\epsilon)^2} \right] \sum_{k=0}^{n} \gamma_k(4Lq; \frac{1}{q}, \ldots, \frac{1}{q}) L_n^{(a)}(t)
\]
\[\text{(48)}\]

It is known that only the term with \(k = 0\) is non-vanishing to the integral of the Laguerre polynomials\textsuperscript{28,64,65} and therefore, the Tsalli and Renyi entropies become Eqs (50) and (51),
\[
T_q[\rho] = \frac{1}{q-1} \left[ 1 - \frac{4\pi C_{N\ell}^2}{(2\epsilon)^2} \right] ^q \sum_{k=0}^{n} \gamma_k(4Lq; \frac{1}{q}, \ldots, \frac{1}{q}) L_n^{(a)}(t)
\]
\[\text{(50)}\]

and
\[
R_q[\rho] = \frac{1}{1-q} \ln \left[ \frac{4\pi C_{N\ell}^2}{(2\epsilon)^2} \right] ^q \sum_{k=0}^{n} \gamma_k(4Lq; \frac{1}{q}, \ldots, \frac{1}{q}) L_n^{(a)}(t)
\]
\[\text{(51)}\]

where
\[
\gamma_k(4Lq; \frac{1}{q}, \ldots, \frac{1}{q}) = \Gamma(4Lq + 1) \left( \frac{2L + N + 1}{N} \right)^{2q} \times F_{\alpha+1}^{(2q+1)}(4Lq + 1, -N, \ldots, -N, 0; 2L + 2, \ldots, 2L + 2, 1; \frac{1}{q}, \ldots, \frac{1}{q})
\]
\[\text{(52)}\]

and \((\alpha)_k = \Gamma(\alpha+n) / \Gamma(\alpha)\).

Consider the case where \(q = 2\) (due to the wide applications of this for both the Tsallis and Renyi entropies). We then make use of Eqs (14) and (26) of Ref. 63 to evaluate the integral of the spherical harmonics. Finally, the Renyi and Tsallis entropies are obtained respectively as Eqs (53) and (54),
\[
T_2[\rho] = 1 - \left[ \sum_{k=0}^{N} C_k(\ell', m') \right] F_2(\ell' + |m'| + k, |m'| + k + 1/2, |m'| + k + 1, 1 \\
+ 2k + 2, 1; \frac{1}{q}, |m'| + k + 1, 1) \]
\[\text{(53)}\]
The Renyi length is plotted against $n$, $D_e$ and $r_e$ in Figs 7-9. The Renyi length increases with $n$ and $r_e$ as shown in Figs 7 and 9 while it decreases with $r_e$ as can be observed from Fig. 8.

**Fisher-Shannon Complexity**

The Fisher-Shannon complexity for the ring-shaped modified Kratzer potential is obtained as Eq. (56),

$$FS = I[\rho] \frac{1}{2\pi e} e^{2S[\rho]}^{1/3} \quad \ldots (56)$$

where $I[\rho]$ is given in Eq. (35) and $S[\rho] = S_{NL}(R) + S_{\text{erm}}(Y)$. $S_{NL}(R)$ is given in Eq. (39) and $S_{\text{erm}}(Y)$ is given in Eq. (41). We have plotted the graphs of the Fisher-Shannon complexity measure against $D_e$ and $r_e$, respectively, in Figs 10 and 11. The Fisher-Shannon complexity initially starts to increase
with $D_e$ (from $D_e = 1$ eV to $D_e = 3$ eV) but starts to decrease monotonically when $D_e = 3$ eV, as shown in Fig. 10. In Fig. 11, the Fisher-Shannon complexity initially is relatively constant with increasing $r_e$ but there is an abrupt increase when $r_e = 20$ Å.

Conclusions

We have obtained the Fisher information measure, Shannon entropy, Renyi entropy, Tsallis entropy and Fisher-Shannon complexity for the ring-shaped modified Kratzer potential. The solution of the Fisher information measure has been obtained exactly. The Shannon entropy has been obtained by making use of the lower bound for the Shannon entropy of the hyperspherical harmonics as obtained by Dehesa et al. It should be noted that (as observed in Ref. 63, the quality of the bound globally decreases with $\ell'$, and it increases with $|m|$ for fixed $\ell'$. We also observed that the angular part of the Shannon entropy has the same value in the states $(n,\ell'm')$ and $(n,\ell'-m')$.

We have made use of entropy of the Laguerre polynomials, Gegenbauer polynomials, and spherical harmonics. We obtained (for $q = 2$) the Tsallis and Renyi entropies, by making use of the linearization formula of Srivastava-Niukkanen for the Laguerre polynomials and spherical harmonics in terms of Lauricella hypergeometric functions.

References

Appendix

Some properties of the associated Laguerre, associated Legendre polynomials and spherical harmonics are given below.

**Associated Laguerre polynomials**

The associated Laguerre polynomials are defined as

\[ L_n^\alpha(x) = \frac{d^n}{dx^n} L_n(x) \]

where

\[ L_n(x) = e^{-x} \frac{d^n}{dx^n} (x^n e^{-x}) \]

is the Laguerre polynomials. The associated Laguerre polynomials are known to satisfy the orthogonality properties\(^{27,28,66}\):

\[ \int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(\alpha+n+1)}{n!} \delta_{nm} \]  \hfill (A1)

\[ \int_0^\infty x^{\alpha-s} L_n^\alpha(x) L_m^\beta(x) dx = \sum_{r=0}^{\min(n,m)} (-1)^{s+r} \binom{s-\alpha}{s-\beta} \binom{s+r}{n-r} \binom{s+r}{m-r} \]  \hfill (A2)

The recurrence relation of the associated Laguerre polynomials is given as\(^{27}\)

\[ xL_n^{\alpha+1}(x) = (k+\alpha+1)L_n^\alpha(x) - (k+1)L_{n+1}^\alpha(x) \]  \hfill (A3)

**Spherical harmonics and associated Legendre polynomials**

The associated Legendre functions \( P_l^m(x) \) are defined in terms of the ordinary Legendre polynomials \( P_l(x) \) as\(^{57-59}\)

\[ P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} (x^2-1)^l \]  \hfill (A4)

Using the Rodrigues' form of the Legendre polynomials, we have (for \( m \geq 0 \))

\[ P_l^m(x) = \frac{(-1)^l}{2^l l!} (1-x^2)^{m/2} \frac{d^m}{dx^m} (x^2-1)^l \]  \hfill (A5)

and for \( m < 0 \),

\[ P_l^{-m}(x) = \frac{(-1)^{-m}}{2^l l!} (1-x^2)^{-m/2} \frac{d^m}{dx^m} (x^2-1)^l \]  \hfill (A6)

(Contd.)
Appendix (Contd.)

The following relation holds between the associated Legendre polynomials and the Legendre Polynomials
\[ P^m_\ell(x) = (-1)^m \frac{(\ell - m)!}{(\ell + m)!} P^m_\ell(x) \] ... (A7)

Also in the literature, the following notations are being used
\[ P^m_\ell(x) = (-1)^m P^m_\ell(x) \]

The associated Legendre polynomials satisfy the following orthonormality condition (for \(0 \leq m \leq \ell\)).
\[ \int_{-1}^{1} P^m_\ell(x) P^n_\ell(x) dx = \frac{2(\ell + m)!}{(2\ell + 1)(\ell - m)!} \delta_{\ell m} \] ... (A8)

The associated Legendre polynomials satisfy the orthogonality relation (for fixed \(\ell\))
\[ \int_{-1}^{1} \frac{P^m_\ell(x) P^n_\ell(x)}{(1 - x^2)^{1/2}} dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{(\ell + m)!}{m(\ell - m)!} & \text{if } m = n \neq 0 \\ \infty & \text{if } m = n = 0 \end{cases} \] ... (A9)

The recurrence relation for the associated Legendre polynomials is given as
\[ (2\ell + 1)x P^m_{\ell+1}(x) = (\ell + m - 1) P^m_\ell(x) + (\ell + m) P^{m+1}_\ell(x) \] ... (A10)

The spherical harmonics of order \(\ell\) are defined as
\[ Y_{\ell m}(\theta, \phi) = C_{\ell m} P^m_\ell(\cos \theta) e^{im\phi} \] ... (A11)

where the normalization constant \(C_{\ell m}\) is given as
\[ C_{\ell m} = \sqrt{\frac{(2\ell + 1)(\ell - m)!}{4\pi (\ell + m)!}} \] ... (A12)

The spherical harmonics fulfill the condition
\[ \int_{-1}^{1} Y^*_{\ell m}(\Omega) Y_{\ell' m'}(\Omega) d\Omega = \delta_{\ell \ell'} \delta_{mm'} \] ... (A13)

where \(d\Omega = \sin \theta d\theta d\phi\).