Evaluation of thermal expansion and compressibility of liquid $^3$He under pressure

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Using the phonon part of the effect Hamiltonian of Mishra and Tripathy (J Low Temp Phys 75 (1989) 79), we have calculated the compressibility and thermal expansion parameter for liquid $^3$He as a function of pressure with fixed temperature using the contact interaction potential and pseudopotential of Aldrich and Pines. With the help of two adjustable parameters one for local field correction effects and other due to damping of zero - sound mode, our calculated values are in good agreement with the experimental data.

1 Introduction

The thermal expansion of liquid $^3$He has been measured a number of times as a function of pressure\(^1\). It was obtained via measurement of entropy of the liquid which relates also with the heat capacity of $^3$He. After the initial disagreement with the heat capacity under pressure and because of low temperature thermometry, now a days measurement of thermal expansion has been performed itself. Recently P R Roach et al.\(^6\), have performed a method to measure the thermal expansion experimentally. They measure the pressure of liquid $^3$He as a function of temperature at a constant molar volume known as $P_r(T)$. The pressure is measured at low temperatures by Straty - Adams capacitative pressure gauge\(^3\). It can resolve at least 10\(^{-1}\) bar over the entire 30 bar range of the experiment. The $^3$He cell contains an acoustic resonator to measure the velocity of sound at the same pressure and molar volume. The compressibility $X_r = 1/pC_v$\(^4\) can be determined from the known values of the density $p$ and the sound velocity $C_v$. From $P_r(T)$, one can obtain $\partial P/\partial T$ and using thermodynamic relation $\alpha = X_r(\partial P/\partial T)$, one can obtain $\alpha$. The advantage of this method over the other\(^5\) method is that no corrections have been made to the pressure determination. No calibration to known liquid is required. Therefore high resolution and high accuracy has been achieved. The result of P R Roach et al.\(^6\) measurement were compared with those of Anderson et al.\(^1\), Lee et al.\(^5\), Bohjosian et al.\(^5\) and Abraham and Osborne\(^5\). Regarding the temperature dependence of thermal expansion $\alpha$, the agreement of Roach et al.\(^6\) data with those of Anderson et al.\(^1\) is very good. There is a systematic difference between their data and those of Bohjosian et al.\(^5\). Although the shape of the curve is the same.

The data of Lee et al.\(^5\) and of Abraham and Osborne\(^5\) disagree with their results and also with others. The data of Brewer and Daunt\(^1\) although have a weaker temperature dependence their results are in fairly good agreement with P R Roach et al.\(^6\).

When thermal expansion $\alpha$, as a function of pressure at a fixed temperature were compared with those of the above workers\(^1\) it was found that the data of Bohjosian et al.\(^5\) lie below their value but have the same shape. The one data point of Anderson et al.\(^1\) agrees fairly with their high pressure data. The data of Abraham and Osborne\(^5\) do not agree with their results at all.

Regarding the theoretical estimation of compressibility and thermal expansion $\alpha$, for liquid $^3$He is concerned there has not been much success in spite of large number of theoretical models available\(^1\). The reason is quite evident that at low temperature one faces serious theoretical difficulties.

Mishra and Tripathy\(^2\) have developed a theoretical model for liquid $^3$He according to which $^3$He is visualised as an interacting system of helium atoms and the quanta of its zero - sound mode. In this
model, they have obtained an effective Hamiltonian for the system by applying a suitable cannonical transformation and the Feynman diagrammatic technique. They have obtained a value of -2.55 K for the average binding energy for particle against the experimental value of (-2.47 ± 0.01) K. This theory also gives rise to a value of 14.1 n m² for the critical wave vector k, which matches with experimental data. In another work, they have been able to reproduce an effective mass of 'He atom (m' = 3 m) which exactly matches with the experimental data of Wheatley and co-workers. They have also evaluated the velocity of zero-sound (204.2 m/s for liquid 'He at P = 0 and T = 0) using a self-consistent method. An exact agreement with the experimental value was obtained by introducing a term into this theory to account for the local field correction to the bare particle-hole propagator beyond the random phase approximation (R P A) in a parametric way. Taking the phonon part of the effective Hamiltonian, Singh and Mishra have determined the sound velocities as a function of pressure to obtain values in good agreement with the experimental data. Singh et al. have also evaluated the temperature variation of the pressure dependent zero-sound velocities in liquid 'He. Very recently Singh and Mishra have determined the pressure-dependent effective mass of the 'He atom in normal liquid 'He and the results are in good agreement with the recent experimental data of Greywall. Tripathy and Mishra have determined the binding energy per nucleon in the ground state of nuclear matter with the same approach using the collective description of the interparticle interactions. Mishra and Mishra have evaluated the energy of zero-sound mode as a function of wave-vector and the results are in good agreement with the recent experimental data. Very recently, they have also presented an evaluation of the Linewidth (damping) of the zero-sound mode in liquid 'He which are also in good agreement with the experimental data.

In this paper, we have presented a method of evaluation of compressibility and thermal expansion αₜ for liquid 'He using Mishra and Tripathy model Hamiltonian. By continuing ourselves to the phonon part of the effective Hamiltonian, an equation for the phonon frequency has been obtained and solved to find out the velocity of zero-sound as a function of pressure. By calculating the density of liquid 'He at different pressure isothermal compressibility has been calculated. By taking the values of (δ/δT) at various pressure from the experimental data, the values of thermal expansion αₜ have been calculated for different fixed temperature. The calculations have been performed for two model interaction potential for liquid 'He in which one is contact interaction potential and other is Pseudopotential whose potential parameter has been given in a paper by Hess and Pines. By using two parameters δ and ξ one for local field corrections and another due to damping of zero-sound mode, we have evaluated the compressibility of liquid 'He as a function of pressure at a fixed temperature. This also enables us to calculate the thermal expansion parameter αₜ as a function of pressure at a fixed temperature. Our evaluated values of these two quantities are in good agreement with the experimental data.

2 Determination of Velocity of Zero-Sound as a Function of Pressure at T = 0

The energy of zero-sound mode as a function of the wave-vector k can be obtained by solving the following equation:

\[
\Omega^2(k) = \Omega_{m0}^2(k) + 2 \Omega_{m0}^\prime \Omega_{m0}^\prime(k)
\]

for \( R \pi \int \frac{1}{k \Omega(k)} \) self-consistently. This is equivalent to looking at the pole of the phonon propagator. The pseudopotential describing the effective interaction between 'He atoms of parallel spin \( \uparrow \uparrow \uparrow \) and antiparallel spin \( \uparrow \downarrow \downarrow \) is given by

\[
\tilde{f}_{\uparrow \uparrow \uparrow} = (1/2) \int d^3 r \{ \tilde{f}_{\uparrow \uparrow \uparrow} + \frac{1}{\pi} \ln \left( \frac{\pi}{2 \xi^2} \right) \}, \quad \text{Exp}(i \vec{k}, \vec{r})
\]

We take the symmetrical part of the potential as

\[
\tilde{f}_{\uparrow \uparrow \uparrow} = (1/2) \int d^3 r \tilde{f}_{\uparrow \uparrow \uparrow} \text{Exp}(i \vec{k}, \vec{r})
\]

We take the symmetrical part of the potential as
where \( n f_{\text{m}} \) and the Fermi energy \( \varepsilon_0 \) are expressed in \( K \).

Now, for the phonon potential expression given in Eq. (5a) in the dimensionless unit

\[
\pi_{\text{ph}}(\tilde{k}, \Omega(\tilde{k})) = \frac{1}{2} \Omega_{\text{ph}}(k) - 1 - \Omega(\tilde{k}) (3/8k^2)
\]

where the above expression has been obtained for \( T = 0 \). Now the velocity of zero-sound \( C_0 \) has been calculated by writing \( \Omega(k) \) in the same form as shown in Eq. (5a). That is we write

\[
\Omega(k) = a_k \sqrt{|k|} = \frac{n f_{\text{m}}}{m}
\]

where \( a_k \) should be equal to \( 2C_0 \). Both \( a_k \) and \( a_\text{c} \) are in the unit of Fermi velocity \( V_F \). Expanding the expression given in Eq. (6) for small \( k \) and then retaining only the terms linear in \( k \), we arrive at the following equation:

\[
a_k = F_1(a_\text{c}, a_\text{s})
\]

where

\[
F_1(a_\text{c}, a_\text{s}) = a_\text{c} \left[ 1 + G_1(a_\text{c}) \right]^{1/2}
\]

\[
G_1(a_\text{c}) = \left[ -1 - (3a_\text{c}^2 / 4) - (3a_\text{s}^2 / 16) \right]
\]

\[
\text{In} \left[ a_\text{c} + 2/a_\text{c} + 2 \right]
\]

For \( a_\text{c} = 2 \), one has to solve the following equation:

\[
a_\text{c} = F_1(a_\text{c}, a_\text{s} = 2, k)
\]

where \( F_1(a_\text{c}, a_\text{s} = 2, k) = a_\text{c} \left[ 1 + G_1(a_\text{c}) \right]^{1/2} \)

and

\[
G_1(a_\text{c}) = \left[ -1 - (3a_\text{c}^2 / 4) - (3a_\text{s}^2 / 16) \right]
\]

\[
\text{In} \left[ a_\text{c} + 2/a_\text{c} + 2 \right]
\]

having

\[
G_1(a_\text{c}) = \left[ -1 - (3a_\text{c}^2 / 4) - (3a_\text{s}^2 / 16) \right]
\]

\[
4 - (3/2) \text{In} k
\]
Now for a given value of $k$ chosen in linear region Eq. (8) is solved self-consistently for a set of values of $a_i$ ($i\neq 2$). One can see that as long as $k$ has a value for which (Eqs 7) is satisfied the value of $F_i(a_i, a)$ for $a_i \neq 2$ does not depend upon $k$. However, at $a_i = 2$ the value of the function $F_i$ varies with $k$. We then make a plot of $a_i, F_i(\frac{a_i}{a})$ versus $a_i$ for fixed value of pressure $P = 0$ for the given two interaction potentials. From the point of intersection of the two curves we find the solution $a_i = F_i(\frac{a_i}{a})$. We then calculate the velocity of zero-sound $C_v = \langle \omega_i \rangle$ for the given potentials for $P = 0$ and $T = 0$.

3 Local Field Correction Effects in the Velocity of Zero-Sound Calculations

Although the self-consistent values of the velocity of zero-sound obtained are close to the experimental data\textsuperscript{25}, we feel that it is an over estimated value, since the local field correction\textsuperscript{26} have been ignored in this theory. One should need a rigorous expression for the particle hole propagator with the proper local field correction included. But its evaluation is very complicated. We assume that the proper polarization correction to $\pi_{\omega}(k, \Omega(k))$ affects the velocity of zero-sound, we take into account its contribution in the present calculation by including a term of the kind $\delta k$ and $\delta$ is an adjustable parameter in this theory. This is considered to be valid for $k = 0$ only. With this non-RPA correction term present Eq. (8b) now assumes the form:

$$a_i = \alpha_i [1 + G, (\alpha_i), \delta]^{1/2}$$

In the case of contact interaction potential\textsuperscript{27} and pseudo-potential\textsuperscript{28} the value of $\delta$ are fixed to be equal to -0.385 and -0.20388 respectively at $P = 0$ to get a good match with the experimental data\textsuperscript{29}.

4 Calculation of Pressure Dependent Zero-Sound Velocities at Finite Temperature and Estimation of Isothermal Compressibility $X_m$ and Thermal Expansion $\alpha_m$

In order to calculate pressure-dependent zerosound velocities at finite temperature, we need the expression for $R\pi_{\omega}(k, \Omega(k))$ for $T \neq 0$ in which case, one writes:

$$R\pi_{\omega}(k, \Omega(k)) = -4 \sum_p F_p(k \cdot \vec{p}) \tilde{r}_p(-k \cdot \vec{p} + \vec{k})$$

where $\tilde{r}_{i,j}(k, \vec{p})$ is given by

$$\tilde{r}_{i,j}(k, \vec{p}) = -\frac{\langle \epsilon_{F_i} - \epsilon_p \rangle}{\Omega^2(k) - \langle \epsilon_{F_i} - \epsilon_p \rangle^2} \times \left[ \frac{\Omega(k) + \Omega_p(k)}{\Omega(k) + \Omega_p(k) + \Omega_{p^2}(k)} \right]^{1/2} \times \left[ \frac{(\Omega(k) + \Omega_p(k))}{(\Omega(k) + \Omega_p(k) + \Omega_{p^2}(k))} \right]^{1/2}$$

with $\beta = \left[ k_B T \right]^{1/2}$ and $\mu$ is the chemical potential of the system. Following the well-known technique dealing with Fermi integrals\textsuperscript{30}, an analytic form for $\pi_{\omega}(k, \Omega(k))$ has been obtained for finite temperatures. We solve the following equation:

$$\Omega^2(k) = \Omega_{p^2}(k) + 2\Omega_{p}(k) \times \left[ \frac{\Omega_p(k)}{\Omega(k) + \Omega_p(k) + \Omega_{p^2}(k)} \right]^{1/2}$$

self-consistently for several pressure for a fixed temperature. After evaluation it turns out that $C_v(T)$ for a particular value of pressure is no different from zero-temperature value. To overcome this difficulty, we try to include the effect of damping into the phonon (zero-sound mode) which amounts to evaluate $\pi_{\omega}(k, \Omega(k))$ for finite temperature in the presence of width in phonon frequency. This is to be justified on the ground that at finite temperatures, one would expect the Landau damping\textsuperscript{31} to be affecting the propagation of zero-sound in the medium. The phonon linewidth $\Gamma_1$ is to be evaluated using the equation:

$$\Gamma_1 = T_1 \pi_{\omega}(k, \Omega_{p^2}(k))$$

In the presence of $\Gamma_1$, $\Omega(k)$ is now determined by solving the equation:

$$\Omega^2(k) = \Omega_{p^2}(k) [1 + \delta + \Gamma_1^2 \Omega_{p^2}(k)]$$

$$\times \sum_p \tilde{r}_p(k, \vec{p}) \tilde{r}_p(-k, \vec{p} + \vec{k})$$

In order to arrive on Eq. (15) we have used:

$$w(k) = (\Omega(k) + i\Gamma_1)$$

Now, choosing $k = 0.01$ (linear region) and temperature $T$ for a particular value for fixed pressure we calculate Eq. (14). It comes out very small for any value of $T < 1500$ mK. In order to match the
experimental values of $C_i(T)$ for different pressure and fixed temperature we need finite value of $\Gamma_k$ at $k = 0.01$. Such a finite value of $\Gamma_k$ at $k \to 0$ can be realized as the experimental data on phonon linewidth $\Gamma_k$ for superfluid 4He approaches a finite constant value for $k \to 0$. For liquid 4He, we presume the situation to be the same. Actually the measurements on the linewidth $\Gamma_k$ of the zero-sound mode in liquid 4He indicates that $\Gamma_k$ decreases with $k$ and attaining a constant value for low $k$.

Without trying to evaluate $\Gamma_k$ from first principle we try to solve Eq. (15) self-consistently by treating $\Gamma_k$ as an adjustable parameter. For zero-temperature, keeping $k = 0.01$ we first solve Eq. (15) after replacing $u \sigma (\overline{p})$ by $\Theta (k_f - \overline{p})$. The expression for $\Omega (k)$ at $T = 0$ in the dimensionless form will be:

$$\Omega^2 (k) = (\delta - 1) - \frac{a}{4} k^2$$

where $C = (4b/\pi)$ for contact interaction potential $^5$ and $C = (4b/\pi)$ for pseudo-potential $^5$.

For small $k$, the Eq. 17 reduces to

$$\Omega^2 (k) = \Gamma_k (0) + a_k k^2 (\delta - 1) - \frac{a}{4} k^2$$

For different pressures the value of $\Gamma_k (0)$ are calculated by taking the experimental value$^5$ of $\Omega (k)$. These have been shown for contact interaction potential$^5$ and pseudo-potential$^5$ in Table 1 and 3 respectively. In order to calculate the value of $C_i(T)$ at a particular temperature and pressure, we now assume that $\Gamma_k (k$ lying under linear region $)$ has a temperature dependence of the form

$$\Gamma_k (T) = \Gamma_k (0) [1 + \eta (T/T_F)]$$

where $\eta$ is an adjustable parameter and $T_F$ is the Fermi-temperature. Now keeping this empirical equation for $\Gamma_k$ Eq. (15) is now solved for $T \neq 0$.

The expression for $\Omega (k)$ at $T \neq 0$ is given by:

$$\Omega^2 (k) = \Gamma_k + a_k k^2 (\delta - 1) - \frac{a}{4} k^2$$

where $C = (4b/\pi)$ for contact interaction potential$^5$ and $C = (4b/\pi)$ for pseudo-potential$^5$. For small $k$, the Eq. 17 reduces to

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For small $k$, the Eq. 17 reduces to

$$\Omega^2 (k) = \Gamma_k + a_k k^2 (\delta - 1) - \frac{a}{4} k^2$$

where $C = (4b/\pi)$ for contact interaction potential$^5$ and $C = (4b/\pi)$ for pseudo-potential$^5$. For small $k$, the Eq. 17 reduces to
The detailed has been discussed in Appendix A.

Now if $k$, and $k^\prime$ denote the compression moduli for the non-interacting and interacting helium atoms in liquid $^4$He then

$$\frac{k^\prime}{k} = \left( \frac{C^2}{C_i^2} \right)$$

...(22)

where $C$, is the actual zero - sound velocity and $C_i$ is the sound velocity for the non-interacting system which is given by $C_i = V_F / \sqrt{3} = V_F$ by the Fermi velocity. Now the isothermal compressibility of non-interacting and interacting system respectively are given by:

$$\chi_i = \frac{1}{\rho C_i^2}$$

...(23)

$$\chi_n = \frac{1}{\rho C_n^2}$$

where $\rho$ is the density of liquid $^4$He. $\rho$ for various pressure has been calculated from data. Now the thermal expansion parameter $\alpha$, for liquid $^4$He is defined as:

$$\alpha_\rho \chi_n \left( \frac{\delta P}{\delta T} \right)$$

...(24)

The values of $(\delta P/ \delta T)$, has been obtained from the Greywall data through interpolation scheme. Using Eq. (24) the value of $\alpha_\rho$ has been calculated for various pressures and fixed temperatures. The results of compressibility $\chi_n$ for two potentials are shown in Tables 2 and 4 respectively.

5 Discussions

In the present work, using the phonon part of the effective Hamiltonian of Mishra and Tripathy we have calculated velocity of zero - sound $C_i(T)$ for various pressures. The compression moduli $K$, and isothermal compressibility $\chi$, have been calculated using the zero-sound at different pressures and
Fig. 1 — Plot of thermal expansion $\alpha_T$ versus pressure $P$ ($P_a$) for two potentials.

Fig. 2 — Plot of thermal expansion $\alpha_T$ versus temperature for the given two potentials.
Fig. 3 — Plot of ratio of compression modules \( \frac{K_o}{K_f} \) versus pressure \( P (P_a) \)

Fig. 4 — Plot of compression modulus \( K_o \) (in Kelvin) versus \( \rho / \rho_s \)

(\( P_a = \text{saturation density} \))
temperatures. We have also calculated the thermal expansion parameter \( \alpha \) for various pressures and fixed temperatures (Figs 1 and 2). We have used two inter-particle potentials namely the contact interaction potential\(^{5}\), and pseudo-potential\(^{6}\) in our calculations. In order to match our calculated values of \( C_v(T) \) for different pressures we have made two corrections in our theory. One is due to local correction\(^{6}\) and other is due to damping of the zero-sound mode. For local field correction effects we have taken an adjustable parameter \( \delta \) to take care of the non-RPA effect. This value of \( \delta \) is different for the two potentials\(^5\)\(^6\) in order to reproduce \( C_v \) at \( P = 0 \). The contact interaction potential which is totally repulsive in nature, the pressure dependent parameter is the scattering length \( b \) of free helium atom, we need \( \delta = -0.385 \) whereas in case of Pseudo - potential which consists of both attractive and repulsive part we need \( \delta = -0.20380 \). In order to obtain the velocity of zero-sound at different temperature and pressure, we have assumed that there exists a damping of zero-sound mode even for small wave-vector at all temperature including zero, the value of \( \Gamma_k \) required by us to reproduce the value of \( C_v \) at \( T = 0 \) has been obtained from a fit with the experimental data\(^5\)\(^6\) using Eq. (18). Knowing this \( \Gamma_k \) \((T) \) is determined through Eq. (19). There is of course not a prior reason on our part to think of such a finite value for \( \Gamma_k \) at \( k = 0 \). There also does not exist any theory, till today, which can reproduce such a value, although the decay of zero-sound mode to two particle-hole pair is reported\(^6\)\(^7\) to give rise to a finite value of \( \Gamma_k \) for low \( k \). Our calculated values of the ratio of compression moduli for the interacting and non-experimental system \((K_{1}/K_{0})\) increases with pressure \((\text{Fig. 3})\) as per experimental observation\(^5\)\(^6\) and other calculations\(^6\) for both the potentials. However, our calculated values of compressibility at three different temperature 100 mk, 300 mk and 600 mk decrease with pressure for both the potentials as physically expected. We make a plot of compression modulus \( K_{0} \) versus \((P/\rho_{e})\), \( \rho_{e} \) is the equilibrium density \((\text{Fig. 4})\). From the graph we find that the compression increases with the increase of particle density. This is what one would physically expect because \( K \propto 1/\chi \). Our evaluated values of thermal expansion parameter \( \alpha \) for these potentials are slightly larger than those of the experimental data\(^5\). Although the values are larger but are in decreasing order as per experimental observations\(^5\)\(^6\) and other measurements\(^5\). We notice a little fluctuation in our values of \( \alpha \) for the intermediate pressure which I think is due to not taking the proper value of \((\delta P/\delta T)\). We have used the data of Greywall\(^5\)\(^7\) and interpolated it for different pressure and temperatures. It is our belief that if we calculate the term \((\delta P/\delta T)\), from our own theory, there would not be any fluctuation in the result of this type. This work is under investigation.

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References

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Appendix - A

Evaluation of $\Omega(k)$ at $T \neq 0$.

We have expression for $\Omega(k)$ at $T \neq 0$ is

$$\Omega^2(k) = \Gamma_k^2 \Omega_{ph}^{(n)}(k) + 2\Omega_{ph}^{(0)}(k)$$

$$\sum_p \tilde{r}_i(\tilde{k}, \tilde{p}) \tilde{r}_j(-\tilde{k}, \tilde{p} + \tilde{k})$$

$$\left[ \frac{(\Omega(k) + \vec{w}_{ph})}{[\Omega(k) + \vec{w}_{ph}]^2 + \Gamma_k^2} \right] \Gamma_k^2$$

$$\left[ \frac{(\Omega(k) - \vec{w}_{ph})}{[\Omega(k) - \vec{w}_{ph}]^2 + \Gamma_k^2} \right] \Gamma_k^2$$

where

$n(p) = \text{Exp}[\beta(\epsilon_p - \mu)]$.

$\beta = (k_b T)^{-1}$

$\mu = \text{chemical potential}$

$\epsilon_p = (p^2 / 2m)$

The coupling constant

$$\nu(\tilde{k}) = \left[ \frac{V(k)}{2\Omega_{ph}^{(0)}(k)} \right]^{1/2} \left( \epsilon_{p+i} - \epsilon_{p-j} \right)$$

...(A3)

where $V(\tilde{k})$ is the interaction potential.

For contact interaction potential

$V(\tilde{k}) = 4\pi b \ h / m$

For pseudo-potential $\tilde{k}$

$V(\tilde{k}) = \int f^{(n)} \left[ \frac{1}{2} \int \tilde{r}_i(\tilde{k}, \tilde{p}) \tilde{r}_j(-\tilde{k}, \tilde{p} + \tilde{k}) \right] (\epsilon_{p+i} - \epsilon_{p-j})$

and

$\Omega_{ph}^{(0)}(k) = \left[ n \int f^{(n)} h^2 / m \right] K$ e

respectively for pseudo-potential and contact interaction potential respectively. Eq. (A3) is made dimensionless by taking

$\beta \rightarrow \beta/\epsilon_c$, $p \rightarrow p/\hbar k_b$

$\mu \rightarrow \mu/\epsilon_c$

$\Omega_{ph}^{(n)}(k) = (a_k K) \epsilon_c$

$$\Gamma_k \rightarrow \Gamma_k \epsilon_c$$

Now changing prime quantity into unprimed quantity such as:

$(p' \rightarrow p, \beta' \rightarrow \beta, \mu' \rightarrow \mu, \text{etc.})$

Eq. (A3) now assumes the form:

$$\Omega^2(k) = \Gamma_k^2 + a_k^2 k^2 (\delta - 1) + C k^2 \int \rho^2 dp \int dx$$

where

$C = (4b \ h / \pi)$ for contact interaction potential

$= 3/2 (n f^{(n)} / \epsilon_c)$ for pseudo-potential

Eq. (A3a) may be adjusted in the form:

$$\Omega^2(k) = \Gamma_k^2 + a_k^2 k^2 (\delta - 1) + C k^2 \int \rho^2 dp \int dx$$

$$[l_1 - l_2] = \frac{1}{e^{\beta(\rho^2 - n^2)} - 1}$$

where

$L_1 = \int \rho^2 (k + 2p x)^2 (\Omega(k) - k^2 - 2p k x)^2 + \Gamma_k^2$ dx

$L_2 = \int \rho^2 (k + 2p x)^2 (\Omega(k) + k^2 + 2p k x)^2 + \Gamma_k^2$ dx

Now after $\rho$ integration

$(I_1 - I_2) = -\frac{1}{4pk^3} \left[ 16pk^3 + (\Omega^2(k) - \Gamma_k^2) \right] \log \left[ \frac{\Omega^2(k) - k^2 - 2p k x^2 + \Gamma_k^2}{\Omega^2(k) - k^2 + 2p k x^2 + \Gamma_k^2} \right]$
\[
4 \Omega(k) \Gamma_i \left\{ \tan^{-1} \left( \frac{\Omega(k) - k^2 + 2 pk}{\Gamma_i} \right) \right\} - \tan^{-1} \left( \frac{\Omega(k) + k^2 - 2 pk}{\Gamma_i} \right)
\]

Putting \( p^2 = \varepsilon \) or \( pdp = 1/2 \, d\varepsilon \)

Eq. A10 reduces to:

\[
\frac{1}{e^{\beta(\varepsilon)\varepsilon + 1}}
\]

This can also be written as:

\[
\Omega^2(k) = \Gamma_i^2 + \alpha^2(k^2(\delta - 1) - C k^2 \int_{\delta}^{\varepsilon} g(\varepsilon) \varepsilon e^{i\varepsilon} d\varepsilon) - \tan^{-1} \left( \frac{\Omega(k) + k^2 + 2 pk}{\Gamma_i} \right)
\]

where \( C'' = (2b/\pi) \) for contact interaction potential\(^7\)

\[
= 3 \left( \frac{\alpha^2}{\varepsilon} \right) \text{ for pseudo-potential}^9\]

\[\text{\ldots (A12)}\]
\[ f(\varepsilon) = \frac{1}{e^{\beta \varepsilon} - 1} \]

\[ g(\varepsilon) = \left[ 4\sqrt{\varepsilon} + \frac{(\Omega^2(k) - \Gamma_e^2)}{4k^2} \right] (A_1 + A_2 - A_1) \]

\[ A_0 = \log \left[ \frac{1}{\Gamma_e} \left( \frac{(\Omega(k) - k^2 - 2k\sqrt{\varepsilon})^2}{1} + \Gamma_e^2 \right) \right] \]

\[ A_1 = \log \left[ \frac{1}{\Gamma_e} \left( \frac{(\Omega(k) + k^2 + 2k\sqrt{\varepsilon})^2}{1} + \Gamma_e^2 \right) \right] \]

\[ A_2 = \log \left[ \frac{1}{\Gamma_e} \left( \frac{(\Omega(k) - k^2 + 2k\sqrt{\varepsilon})^2}{1} + \Gamma_e^2 \right) \right] \]

\[ P_1 = \tan^{-1} \left( \frac{\Omega(k) - k^2 + 2k\sqrt{\varepsilon}}{\Gamma_e} \right) \]

\[ P_2 = \tan^{-1} \left( \frac{\Omega(k) - k^2 - 2k\sqrt{\varepsilon}}{\Gamma_e} \right) \]

\[ P_3 = \tan^{-1} \left( \frac{\Omega(k) + k^2 - 2k\sqrt{\varepsilon}}{\Gamma_e} \right) \]

\[ P_4 = \tan^{-1} \left( \frac{\Omega(k) + k^2 + 2k\sqrt{\varepsilon}}{\Gamma_e} \right) \]

Using the techniques of well-known Fermi integrals, Eq. (13) can be written in the form:

\[ \Omega^2(k) = \Gamma_e^2 + \frac{3}{2} k^2 (\Delta - 1) - C k^2 \int_{-\infty}^{\infty} g(\varepsilon) d\varepsilon \]

\[ + \left( \frac{\pi^2}{\sigma} \right) \left( \frac{T}{T_r} \right)^2 \int_{-\infty}^{\infty} \left[ g(\varepsilon) \frac{d g(\varepsilon)}{d \varepsilon} \right] \]

Now,

\[ \frac{d g(\varepsilon)}{d \varepsilon} = \left[ \frac{\Omega(k) - k^2}{4k^2} \right] \frac{\Gamma_e^2}{(\Omega(k) - k^2 + 2k\sqrt{\varepsilon})^2 + \Gamma_e^2} \]

Similarly,\n
\[ I_y = \frac{d}{d \varepsilon} \left( 4\sqrt{\varepsilon} = 2l \sqrt{\varepsilon} \right) \]

\[ I_0 = \frac{d}{d \varepsilon} \left[ \log \left[ \frac{(\Omega(k) - k^2 - 2k\sqrt{\varepsilon})^2 + \Gamma_e^2}{(\Omega(k) - k^2 - 2k\sqrt{\varepsilon})^2 + \Gamma_e^2} \right] \right] \]

Similarly,\n
\[ I_y = \frac{d}{d \varepsilon} \left( 2k \left\{ \frac{\Omega(k) - k^2 + 2k\sqrt{\varepsilon}}{(\Omega(k) - k^2 + 2k\sqrt{\varepsilon})^2 + \Gamma_e^2} \right\} \right) \]

Now,\n
\[ I_y = \frac{d}{d \varepsilon} \left( \tan^{-1} \left[ \frac{\Omega(k) - k^2 + 2k\sqrt{\varepsilon}}{\Gamma_e} \right] \right) \]

We know that\n
\[ d/dx \tan^{-1}(x/a) = a/(a^2 + x^2) \]

or\n
\[ d/dx \tan^{-1}(x) = 1/(1 + x^2) \]

Let \( t = \left[ \frac{(\Omega(k) - k^2 + 2k\sqrt{\varepsilon})}{\Gamma_e} \right] \)

So,\n
\[ I_y = \frac{d}{d \varepsilon} \left( \tan^{-1} t \right) = \frac{d}{d t} \left( \tan^{-1} t \right) \times \frac{dt}{d \varepsilon} \]

\[ = \frac{1}{(1 + t^2)} \times 2k \left( \frac{1}{2\sqrt{\varepsilon}} \right) \]

\[ = \frac{k}{\sqrt{\varepsilon}} \left[ \Gamma_e^2 + (\Omega(k) - k^2 + 2k\sqrt{\varepsilon})^2 \right] \]

Similarly,\n
\[ I_y = -\frac{k}{\sqrt{\varepsilon}} \left[ \left( \Omega(k) - k^2 + 2k\sqrt{\varepsilon} \right)^2 + \Gamma_e^2 \right] \]
\[
\begin{align*}
I_1 &= -\frac{k}{\sqrt{e}} \left\{ \frac{\Gamma_i^2}{(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2} \right\} \\
I_2 &= \frac{k}{\sqrt{e}} \left\{ \frac{\Gamma_i^2}{(\Omega(k) + k^2 + 2k\sqrt{e})^2 + \Gamma_i^2} \right\}
\end{align*}
\]

\[
\frac{dg(e)}{de} = \left[ 2 + \frac{\Omega^2(k) - \Gamma_i^2}{2k^2} \right] - \frac{(\Omega(k) + k^2 - 2k\sqrt{e})}{[(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2]} \\
+ \frac{(\Omega(k) + k^2 - 2k\sqrt{e})}{[(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2]}
\]

\[
\Omega^2(k) = \Gamma_i^2 + \epsilon^2 k^2 (\delta - 1) - \epsilon^2 k^2 \left\{ \int_0^d \text{d}e \left\{ 4\sqrt{e} + \frac{(\Omega^2(k) - \Gamma_i^2)}{4k^2} \right\} \log_e \left\{ \frac{(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2}{(\Omega(k) + k^2 + 2k\sqrt{e})^2 + \Gamma_i^2} \right\} + \left\{ \frac{(\Omega^2(k) - \Gamma_i^2)}{k^2} \right\} \left\{ \frac{\Omega(k) + k^2 - 2k\sqrt{e}}{\Gamma_i} \right\} \right\}
\]

\[
\frac{dg(e)}{de} = \left[ 2 + \frac{\Omega^2(k) - \Gamma_i^2}{2k^2} \right] - \frac{(\Omega(k) + k^2 - 2k\sqrt{e})}{[(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2]} \\
+ \frac{(\Omega(k) + k^2 - 2k\sqrt{e})}{[(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2]}
\]

\[
\Omega^2(k) = \Gamma_i^2 + \epsilon^2 k^2 (\delta - 1) - \epsilon^2 k^2 \left\{ \int_0^d \text{d}e \left\{ 4\sqrt{e} + \frac{(\Omega^2(k) - \Gamma_i^2)}{4k^2} \right\} \log_e \left\{ \frac{(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2}{(\Omega(k) + k^2 + 2k\sqrt{e})^2 + \Gamma_i^2} \right\} + \left\{ \frac{(\Omega^2(k) - \Gamma_i^2)}{k^2} \right\} \left\{ \frac{\Omega(k) + k^2 - 2k\sqrt{e}}{\Gamma_i} \right\} \right\}
\]

So finally we have at \( T \neq 0 \)

\[
\frac{dg(e)}{de} = \left[ 2 + \frac{\Omega^2(k) - \Gamma_i^2}{2k^2} \right] - \frac{(\Omega(k) + k^2 - 2k\sqrt{e})}{[(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2]} \\
+ \frac{(\Omega(k) + k^2 - 2k\sqrt{e})}{[(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2]}
\]

\[
\Omega^2(k) = \Gamma_i^2 + \epsilon^2 k^2 (\delta - 1) - \epsilon^2 k^2 \left\{ \int_0^d \text{d}e \left\{ 4\sqrt{e} + \frac{(\Omega^2(k) - \Gamma_i^2)}{4k^2} \right\} \log_e \left\{ \frac{(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2}{(\Omega(k) + k^2 + 2k\sqrt{e})^2 + \Gamma_i^2} \right\} + \left\{ \frac{(\Omega^2(k) - \Gamma_i^2)}{k^2} \right\} \left\{ \frac{\Omega(k) + k^2 - 2k\sqrt{e}}{\Gamma_i} \right\} \right\}
\]

\[
\frac{dg(e)}{de} = \left[ 2 + \frac{\Omega^2(k) - \Gamma_i^2}{2k^2} \right] - \frac{(\Omega(k) + k^2 - 2k\sqrt{e})}{[(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2]} \\
+ \frac{(\Omega(k) + k^2 - 2k\sqrt{e})}{[(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2]}
\]

\[
\Omega^2(k) = \Gamma_i^2 + \epsilon^2 k^2 (\delta - 1) - \epsilon^2 k^2 \left\{ \int_0^d \text{d}e \left\{ 4\sqrt{e} + \frac{(\Omega^2(k) - \Gamma_i^2)}{4k^2} \right\} \log_e \left\{ \frac{(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2}{(\Omega(k) + k^2 + 2k\sqrt{e})^2 + \Gamma_i^2} \right\} + \left\{ \frac{(\Omega^2(k) - \Gamma_i^2)}{k^2} \right\} \left\{ \frac{\Omega(k) + k^2 - 2k\sqrt{e}}{\Gamma_i} \right\} \right\}
\]

\[
\frac{dg(e)}{de} = \left[ 2 + \frac{\Omega^2(k) - \Gamma_i^2}{2k^2} \right] - \frac{(\Omega(k) + k^2 - 2k\sqrt{e})}{[(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2]} \\
+ \frac{(\Omega(k) + k^2 - 2k\sqrt{e})}{[(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2]}
\]

\[
\Omega^2(k) = \Gamma_i^2 + \epsilon^2 k^2 (\delta - 1) - \epsilon^2 k^2 \left\{ \int_0^d \text{d}e \left\{ 4\sqrt{e} + \frac{(\Omega^2(k) - \Gamma_i^2)}{4k^2} \right\} \log_e \left\{ \frac{(\Omega(k) + k^2 - 2k\sqrt{e})^2 + \Gamma_i^2}{(\Omega(k) + k^2 + 2k\sqrt{e})^2 + \Gamma_i^2} \right\} + \left\{ \frac{(\Omega^2(k) - \Gamma_i^2)}{k^2} \right\} \left\{ \frac{\Omega(k) + k^2 - 2k\sqrt{e}}{\Gamma_i} \right\} \right\}
\]
where

\[ \mu = \mu_0 \left[ 1 - \left( \frac{\pi}{12} \right) \left( \frac{T}{T_r} \right) \right] \]

and

\[ \Gamma_\alpha(T) = \Gamma_\alpha(0) \left[ 1 + \left( \frac{T}{T_r} \right) \right] \]

\[ \left[ \frac{(\Omega(k) - k - 2k) + \Gamma_\alpha(T)}{(\Omega(k) + k - 2k) + \Gamma_\alpha(T)} \right] \]