Moduli space for dyons

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Analyzing the non-Abelian gauge theory of dyon the expression for dyonic mass has been derived and connection of moduli space of monopoles with normalized bosonic and fermionic spaces is established. Supersymmetrizing this theory of dyons it has been demonstrated that the modification in supersymmetric algebra due to central charges leads to partial breaking of supersymmetries with the unbroken supersymmetry pairing the bosonic and fermionic zero modes. The moduli space has been constructed in non-Abelian gauge theory of dyons and its kinematics have been investigated.

I Introduction

Monopoles and dyons are the intrinsic parts of all current grand unified theories with enormous potential importance in connection with their role in catalyzing proton decay, the quark confinement problem and C.P violation etc. The dyon-fermion dynamics has been worked out by various authors and it has been shown that the nature of the dyons is strongly perturbed by fermionic sector which couples with them.

In supersymmetric grand unified theories the monopole ground state, in general carries fractional electric charge as well as color hypercharge. This is manifestation of fermion fractionalization with the axial anomaly effect properly taken into account. It is known that the monopole states which are connected by Jackiw-Rebbi zero mode operators must be embedded into fundamental multiplet of \( N = 1 \) supersymmetric Georgi-Glashow model and then the Jackiw-Rebbi zero modes exactly coincide with supersymmetric zero modes. If the Jackiw-Rebbi zero modes exist independently of the supersymmetric zero-modes, the dyonic states are enriched by the further degeneracy with different charge and spin states and consequently the supersymmetry becomes very much complicated. The explicit demonstration of Jackiw-Rebbi zero modes is extremely difficult but some efforts have been made in this direction by calculating one loop quantum corrections to the mass of monopole in a non-Abelian supersymmetric gauge field theory. The existence of dyonic super-multiplets has been demonstrated and it has been shown that the dyons in \( N = 1 \) supersymmetric theory can be consistently represented by super-multiplets containing spin-0 and spin-1/2 states. In the present paper, we have analysed the extended structure of non-Abelian dyons by calculating the quantized electromagnetic fields in the interior region of dyons and supersymmetrized the dyonic Lagrangian density introducing Dirac spinor and pseudo scalar into it. Analysing the non-Abelian gauge theory of dyon, the expression for dyonic mass has been derived and connection of moduli space of monopoles with normalized bosonic and fermionic spaces is established. Generalizing the theory of dyons to \( N = 2 \) supersymmetric theory has been demonstrated that the modification in supersymmetric algebra due to central charges leads to partial breaking of supersymmetries with the unbroken supersymmetry pairing the bosonic and fermionic zero modes. Investigating the kinematics of resulting moduli space, the formulae for dyonic mass and dyonic charge have been obtained in classical as well as in quantum spaces. The transformations of electric and magnetic charge numbers of dyons in quantum moduli space have been obtained under the duality group \( SL(2,Z) \) and it has also been demonstrated that electric and magnetic duality is not possessed in \( N = 2 \) supersymmetric theory. The curves of marginal stability have been obtained in the weak as well as in strong coupling regions.

2 Non-Abelian Theory of Dyons

External four space and \( n \)-dimensional internal group space is contained in a general non-abelian gauge theory of dyons. The field equations and equation of motion for dyons preserve the invariance under the local non-Abelian gauge transformations.
\[ \psi \to \psi' = S^{-1} \psi \quad \ldots (2.1) \]

where \( S \) is a local group element of \( \text{SU}(2) \). Here the local gauge theory makes the presence of the field necessary.

A generalized potential \( V_\mu \) is introduced here as:

\[ V_\mu = A_\mu + iB_\mu \]

\( A_\mu \) corresponds to electric charge and \( B_\mu \) the magnetic charge and the derivative of \( \psi \) is defined as:

\[ V_\mu (\phi) = (\partial_\mu - i q^\mu \epsilon^{\mu
u\rho} \cdot V_{\mu\nu\rho} \phi) \quad \ldots (2.2) \]

where, \( \mu = 0,1,2,3 \) are the indices representing degrees of freedom in external space and \( V_\mu \) in the internal space is the generalized four potential given by:

\[ V_{\mu} = A_{\mu} + iB_{\mu} \quad (\mu = 1,2,3) \]

where \( A_{\mu} \) and \( B_{\mu} \) are the electric and magnetic four-potentials associated with dyons carrying the generalized charge

\[ q = e + ig \quad \ldots (2.3) \]

in terms of electric and magnetic charges \( e \) and \( g \) respectively.

Generalized electromagnetic field tensor can be written as:

\[ G^\mu_\nu = V^\mu_\nu - V^\mu_\nu + i q^\mu \epsilon^{\mu
u\rho} \cdot V_{\mu\nu\rho} \quad \ldots (2.4) \]

where \( \epsilon^{\mu\nu\rho} \) is the structure constant.

In the internal two dimensional complex space, introduced at each point of Minkowski space-time; charge field \( \psi \) in \( \text{SU}(2) \) is replace by \( \exp [i\Lambda^\mu (x)] \psi (x) \) in \( \text{SU}(2) \times \text{U}(1) \) where \( \Lambda^\mu (x) \) is the phase factor. Then the basic spinors of internal space are equivalently acted upon by:

\[ S(x) = S(x) \exp [-i\Lambda^\mu (x)] \]

where \( S(x) \) is the local group element of \( \text{SU}(2) \).

Under the gauge transformation, potential \( V_\mu \) and the matrix field tensor \( G_{\mu\nu} \) transform as follows:

\[ V'_\mu = S^{-1} V_\mu S - S^{-1}\partial_\mu S \]

and

\[ G'_{\mu\nu} = S^{-1} G_{\mu\nu} S \quad \ldots (2.5) \]

In terms of the matrices \( T_\alpha \), which are the three generators of Yang-Mills theory\(^{24}\), above four vector \( V_\mu \) and field tensor \( G_{\mu\nu} \) can be expressed as:

\[ V_\mu = V^{\mu}_\sigma T_\sigma \]

\[ G_{\mu\nu} = G^{\mu\nu}_\sigma T_\sigma \quad \ldots (2.6) \]

The matrices \( T_\alpha \), the infinitesimal generators of the group \( \text{SU}(2) \), satisfy the commutation relation

\[ [T_\alpha, T_\beta] = i\epsilon_{\alpha\beta\gamma} T_\gamma \quad \ldots (2.7) \]

Here \( \epsilon_{\alpha\beta\gamma} \) is the structure constant in the internal group.

Eqn. (2.4) can be written as:

\[ G^\mu_\nu = \partial_\mu V_\nu - \partial_\nu V_\mu + i q^\mu \epsilon^{\mu
nu\rho} \cdot V_{\mu\nu\rho} \cdot V_\nu \quad \ldots (2.8) \]

with its covariant derivative given by:

\[ \nabla^\mu G^\mu_\nu = \partial^\mu G^\mu_\nu + i q^\mu \epsilon^{\mu
nu\rho} \cdot V_{\mu\nu\rho} \cdot G_\mu \]

\[ = J^\mu_\nu + i q^\mu \epsilon^{\mu
nu\rho} \cdot V_{\mu\nu\rho} \cdot G_\mu = J^\mu_\nu \quad \ldots (2.9) \]

where

\[ J^\mu_\nu = \partial^\mu \phi^\nu \phi - \partial^\nu \phi^\mu \phi \quad \ldots (2.9a) \]

represents the generalized linear electric and magnetic current \( J^\mu_\nu \) and \( k^\mu_\nu \) associated with dyons\(^{15}\). The Noetherian current \( J^\mu_\nu \) is manifestly conserved while generalized non-Abelian current \( J^\mu_\nu \) is not so but it satisfies following condition of gauge covariance

\[ \nabla_{\mu} J^\mu_\nu = 0 \quad \ldots (2.9b) \]

The suitable Lagrangian density for a spontaneously broken non-Abelian gauge theory has the following form\(^{16}\)

\[ L = \frac{1}{4} G^\mu_\nu G^\nu_\rho (D_\rho \phi_\sigma) (D^\mu \phi_\sigma) - V(\phi) \quad \ldots (2.10) \]

where the potential term \( V(\phi) \) is necessary for the local theory, \( \phi \) is the vacuum energy which is given by the Higg's field and

\[ (D_\rho \phi_\sigma) = \partial_\rho \phi_\sigma + q^\rho \epsilon^{\rho\mu\nu} \cdot V_{\mu\nu} \phi_\sigma \quad \ldots (2.11) \]

\[ V(\phi) = \frac{1}{4} (\phi^\rho \phi_\sigma)^2 - \frac{1}{2} \phi^2 - \frac{1}{2} \phi^2 \quad \ldots (2.12) \]

with \( v^2 = <\phi>^2 \)

which determines the vacuum expectation value of triplet Higg's field \( \phi^0 \).

Using this Lagrangian density we have computed\(^{16}\) the electromagnetic fields and the resulting angular momentum of dyons by imposing the conditions of Julia and Zee\(^{17}\).

\[ V_\mu = \frac{1}{q^\mu} \left[ \frac{1}{1} K(r) - 1 \right] \]

\[ V_\nu = \frac{1}{q^\nu} \left[ \frac{1}{1} K(r) - 1 \right] \]

\[ \phi_\nu = \frac{1}{q^\nu} \left[ \frac{1}{1} K(r) - 1 \right] \]

which satisfy the following equations:

\[ r^2 H^\nu (r) = 2 HK^2 \]

\[ r^2 J^\nu (r) = 2 JK^2 \]
\[ r^2 K'(r) = K(K^2 - 1) + K(H^2 - J^2) \]  
\[ \text{Solution of these equations may be written as follows:} \]
\[ J(r) = \alpha \phi(r) \]
\[ H(r) = \beta \phi(r) \]
\[ K(r) = \frac{C_r}{\sinh C r} \]

where \( \beta^2 - \alpha^2 = 1 \) and \( \alpha, \beta \) and \( C \) are constants.

In Prasad-Sommerfield limit, taking \( V(\phi) = 0 \), the vacuum expectation value is:

\[ \langle \phi | \phi \rangle = \nu \neq 0 \]

This limit the dyon solutions have the lowest possible energy for given electric and magnetic charges \( e \) and \( g \) respectively.

We get the following expressions for Higg's mass:

\[ M = \nu (e^2 + g^2)^{1/2} = v|\phi| \]

when the electric and magnetic fields associated with dyon obey the first order equations:

\[ E_a^a = \partial_a V_a^a - \partial_a V^a + q^a \epsilon^{abc} V_{bc} V_{ac} \]

\[ = \left( \frac{r_0}{l_1} \right) \frac{\partial}{\partial r} \left[ \frac{K(r) - 1}{r} \right] + \left[ \frac{K(r) - 1}{r^2} \right] r_0 \]

\[ = (D, \phi) \gamma^a \sin \alpha \]

\[ B_a^a = \frac{1}{2} \epsilon_{ijk} G_{jk}^{ab}, (i \neq j \neq k) \]

\[ = \left( \frac{r_0}{l_1} \right) \frac{\partial}{\partial r} \left[ \frac{K(r) - 1}{r} \right] \]

\[ = \left( \frac{r_0}{l_1} \right) \frac{\partial}{\partial p_l} \left[ \frac{K(r) - 1}{r} \right] r_0 - \left. \frac{r_0}{l_1} \right| \frac{K(r) - 1}{r} \]

\[ = (D, \phi) \gamma^a \cos \alpha \]

and

\[ D_{\alpha}(\phi) = a^{\alpha}(\phi) \]

where

\[ \alpha = \tan^{-1} \frac{e}{g} \]

3 Supersymmetrization

We supersymmetrize the dyonic Lagrangian given by Eq. (2.10) in the following form by introducing the fermionic terms:

\[ L = -\frac{1}{4} G^{\mu \nu} G_{\mu \nu} + \frac{1}{2} (D_{\mu} \phi) (D^{\mu} \phi)^* + \]

\[ \frac{1}{2} (D_{\mu} P_{\nu}) (D^{\mu} P_{\nu}) - V(\phi, P) + i \bar{\psi} \gamma^a (D_{\mu} \psi) + \]

\[ - \frac{1}{2} (q^a \epsilon_{a b c} \bar{\psi} \gamma^b \phi^* - q^a \epsilon_{a b c} \bar{\psi} \gamma^b \phi^*) - q^a \epsilon_{a b c} \bar{\psi} \gamma^b P_{\mu} \psi^* P_{\mu} \]

where \( \psi \) is introduced as Dirac spinor and \( P \) as Pseudo-scalar.

A very important property of this model is that the vacuum energy is independent of the values of \( \phi \) and \( P \) in certain directions in field space. As long as \( \phi \) and \( P \) commute the vacuum energy is classically zero. \( \phi \) and \( P \) may have non-zero vacuum expectation values spontaneously breaking some of the gauge symmetries. Since \( \phi \) and \( P \) commute in SU(2), they must be proportional to each other. By preserving a chiral rotation we can assume that \( \langle P \rangle = 0 \).

Under the supersymmetric transformations, the potential field, Higg's field, Pseudo-scalar and wave function included in the Lagrangian, transform as follows:

\[ \delta \phi = i (\alpha \gamma^a \psi - \psi \gamma^a \alpha) \]

\[ \delta \psi = i (\alpha \gamma^a \alpha - \psi \gamma^a \psi) \]

\[ \delta P = \psi \gamma^a \gamma_5 \psi \gamma^a \alpha \]

where \( \alpha \) is a constant anticommuting Dirac spinor.

The potential term \( \Delta \psi \psi \) in the supersymmetrized Lagrangian breaks the supersymmetry yielding dyonic solutions for the following expectation values of \( \phi \) and \( P \):

\[ \langle \phi \rangle = \nu = \langle \alpha \phi | \phi \rangle, \text{ where } \phi = \left( \begin{array}{c} a \\ \nu \end{array} \right) \]

\[ \langle P \rangle = 0 = \langle \alpha \phi | P \rangle \]

Thus the splitting occurs here into two supersymmetric multiplets. First is massless and contains \( \psi, \phi \), and the second contains non-zero mass \( M = \nu 1 \), and contains the fields \( V_{\mu} \), \( S^\alpha \), \( \psi^c \). In supersymmetric theory described by Lagrangian (3.1) the dyons must exist in the form of bosons together with their fermionic partners. In the usual supersymmetric theory one has the following supersymmetric algebra in terms of supersymmetric central charges:

\[ (Q_\alpha, Q_\beta) = (Q_\alpha, Q_\beta) = 0 \]

\[ (Q_\alpha, Q_\beta) = 2 \sigma^a_{\alpha \beta} P_\lambda \]

and the dyonic masses are changed due to the presence of supersymmetric algebra.

Electric and magnetic charges are:
where the gauge parameter \( \Lambda(x) \) is restricted to be independent of \( x^4 \), and
\[
D_\mu W_\mu - D_\mu W_\nu = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} D^\rho W^\sigma
\]
This is the affine connection condition in the Yang-Mills field. Due to the presence of central charges, the supersymmetric algebra changes as:
\[
\{(\bar{Q},Q)\} = 2\gamma_\mu P^\mu - 2Q e - 2 i \gamma_5 Q M
\]
where \( Q_\nu = v_\nu \) and \( Q_M = v_M \)

The fermionic zero modes are time independent C-number solutions to the Dirac equation in the presence of monopoles. So the usual Dirac equation \( \gamma^\mu D_\mu \psi = 0 \) is now modified as:
\[
\gamma^\mu D_\mu \psi - i \gamma_5 \frac{Q_\nu}{Q_M} \psi = 0 \quad \ldots (3.10)
\]
due to the presence of central charges.

If supersymmetry breaks, two independent zero modes are created satisfying
\[
\Gamma_3 \psi = -\psi'
\]
where
\[
\Gamma_3 = -i \gamma_\nu \gamma_5 , \Gamma_4 = 1 , \Gamma_5 = \gamma_5
\]
These are the fermionic Goldstone modes corresponding to the broken supersymmetry. In the one monopole sector there are only fermionic zero modes. In the multi-monopole sector there are other fermionic zero modes also. According to Callias index theorem \( 22 \) in the \( k \)-monopole sector there are \( 2k \) fermionic zero modes.

Let us introduce Hermitian-Euclidean gamma matrices via:
\[
\Gamma_1 = \gamma_\nu \Gamma_5 ; \Gamma_3 = \gamma_6
\]
satisfying:
\[
[\Gamma_\mu , \Gamma_\nu ] = 2 \delta_\mu \nu \Gamma_5 \gamma_\sigma
\]
We impose the supersymmetric invariant restrictions on the equation of motion.

In case of all the fields as time independent, the equations of motion are:
\[
F_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma} ; \quad \Gamma_\nu D^\nu \psi = 0 ; \quad D_\mu D_\nu - P^\nu = i [ \psi' , \psi ] \quad \ldots (3.11)
\]
and unbroken supersymmetric transformations are:
\[
\delta W_\mu = i \alpha \cdot \Gamma_\mu \psi - i \gamma_5 \gamma_\sigma \alpha \cdot \Gamma_\sigma W_\mu
\]
\[
\delta \psi' = -2 \Gamma_\nu D^\nu P^\nu \alpha ; \quad \delta P^\nu = 0 \quad \ldots (3.12)
\]
Equation of motion (3.11) are covariant with respect to these transformations if \( \alpha \) is a Grassmann odd spinor.
If \( \alpha_\nu = \epsilon_\nu \) is a C-number spinor, the Dirac equation in (3.11) is not covariant. Thus for each fermionic zero
mode satisfying \( \Gamma \psi = -\psi \), we have a bosonic zero mode satisfying equation:
\[
\delta W_ \mu = i \epsilon^ \mu \Gamma_ \mu \psi - i \psi \Gamma_ \mu \epsilon^ \mu \quad \text{...(3.13)}
\]

The effective action\(^{25,26}\) governing the low energy dynamics of monopoles of \( N = 2 \) supersymmetric Yang-Mills is given by \( N = 4 \) supersymmetric quantum mechanics based on the moduli space of static BPS monopoles. Similarly to study the dynamics of the monopoles of \( N = 4 \) supersymmetric Yang-Mills, we need \( N = 8 \) supersymmetric quantum mechanics based on moduli space \( M_L \).

Since the monopole configuration breaks only half of the supersymmetries of the field theory, the condition can be written as:
\[
\alpha_+ = \frac{1}{2} | 1 + \Gamma_\mu | \quad \text{...(3.14)}
\]
where \( \alpha_+ \) is the parameter of unbroken and \( \alpha_- \) is the parameter of broken symmetry.

Though the electric-magnetic duality
\[
e \Leftrightarrow -e; E \rightarrow H; H \rightarrow E \quad \text{...(3.15)}
\]
is satisfied in \( N = 1 \) supersymmetric theory\(^{26}\). Such duality condition is violated in \( N = 2 \) supersymmetric theory and maintained again in \( N = 4 \) supersymmetric theory.

### 4 Moduli Space Associated With Dyons

In supersymmetrising Lagrangian (3.1) all fields are in adjoint representation of SU(2) and \( V(\phi, P) \) is the classical potential represented by:
\[
V(\phi, P) = \frac{1}{\| \phi \|^4} t r (\phi \cdot P) \quad \text{...(4.1)}
\]

In the classical potential (4.1) of \( N = 2 \) theory without hyper- multiplet (i.e., without quarks), let us set \( \phi = \phi^+ \) such that:
\[
V(\phi, P) = V(\phi) = \frac{1}{\| \phi \|^4} t r (\phi \cdot \phi^+) \quad \text{...(4.2)}
\]
where \( \phi^+ \) is the gauge coupling constant of the underlying microscopic theory. As long as \( \phi \) and \( \phi^+ \) commute the scalar potential \( V(\phi) = 0 \) even for non-vanishing expectation value of \( \phi \) given by (2.16) which spontaneously breaks SU(2) to U(1) showing that the theory has a continuum of gauge inequivalent vacua called the classical moduli space parametrized by
\[
\eta = t r \phi^+ = \frac{1}{2} \sqrt{2} \quad \text{...(4.3)}
\]
where for SU(2) gauge group, we have set \( \phi = \frac{1}{2} \sqrt{2} \sigma^3 \) with:
\[
\sigma^3 = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

For \( \eta \neq 0 \), the gauge symmetry is broken to U(1) and the global \( Z_8 \) symmetry is broken to \( Z_4 \). The parameter \( \eta \) given by (4.3) is a good local co-ordinate of the classical moduli space. Each point in this moduli space represents a different theory. For Lagrangian remains invariant under these transformations, \( \eta \neq 0 \), the charged multiplets acquire the mass:
\[
M = \sqrt{2} \sqrt{v} | q | \quad \text{...(4.4)}
\]
and SU(2) is spontaneously broken to U(1), while for \( \eta = 0 \) the full SU(2) is restored. Away from the origin we can integrate out massive multiplets and obtain a low-energy effective theory which depends only on photon multiplet. For energy \( < \phi > \), the low energy effective Lagrangian containing a single \( N = 2 \) vector multiplet \( v \), is expressed in terms of a holomorphic function:
\[
F(\eta, v) = \frac{1}{2} t r (v^2) \quad \text{...(4.5)}
\]
where \( t r (v) \) is effective coupling in the vacuum parametrized by \( v \). The metric\(^{26}\) is:
\[
d s^2 = \Im \frac{\partial^2 F(\eta, v)}{\partial \eta^2} d \tau d \eta = \Im \frac{\partial t r (v)}{\partial \eta} \eta d \eta d v \quad \text{...(4.6)}
\]
where
\[
\tau(v) = \frac{\partial^2 F(\eta, v)}{\partial \eta^2} \quad \text{...(4.7)}
\]

Defining electric and magnetic charge numbers \( n_e \) and \( n_m \) respectively as follows in the classical moduli space:
\[
n_e = \frac{e}{\sqrt{2}} \quad \text{and} \quad n_m = \frac{e}{8 \pi} \cdot \frac{g}{\sqrt{2}} \quad \text{...(4.8)}
\]
The dyonic charge may be written as:
\[
g = (n_e, n_m) = \sqrt{2} \left( n_m - \frac{8 \pi i}{g^2} n_e \right) \quad \text{...(4.9)}
\]
and the dyonic mass given by inequality (3.6) may be written in the following form for BPS states:
\[
M = \sqrt{2} \sqrt{v} \left| n_e - \frac{8 \pi i}{g^2} n_m \right| = \sqrt{2} \sqrt{v} | n_e - \tau | n_m \quad \text{...(4.10)}
\]
where
\[
\tau = \frac{8 \pi i}{g^2} \quad \text{...(4.11)}
\]
Identifying this \( \tau_b \) as the effective coupling constant \( \tau(v) \) of Eq. (4.7), we get:

\[
F(v) = \frac{4\pi v^2}{r^2} + C v + D
\]

where \( C \) and \( D \) are the constants of integration. Imposing the following condition on the holomorphic function:

\[
\frac{\partial F(v)}{\partial v} = 0
\]

we get \( C = 0 \), then Eq. (4.9) becomes:

\[
F(v) = \frac{4\pi v^2}{r^2} + D
\]

The curve separating weak and strong coupling regions in quantum moduli space is the curve of marginal stability along which dyonic charge is given as:

\[
q = \sqrt{2} (n_e - \tau, n_m) = (n_e, n_m)
\]

Setting \( v = a \) and \( \frac{\partial F}{\partial a} = a_0 \rightarrow \tau(a) = \frac{\partial a_0}{\partial a} \)

We may write Eq. (4.6) for metric of classical moduli space in the following form:

\[
ds^2 = 1 \text{m} \frac{\partial F(v)}{\partial v} dv d\bar{v} = \text{Im} \frac{\partial a_0}{\partial a} da d\bar{a}.
\]

\[
= \text{Im} d a_0 d\bar{a} = -i \left( d a_0 d\bar{a} - d a d a_0 \right)
\]

which is symmetrical in \( a \) and \( a_0 \). In general \( a \) and \( a_0 \) are the functions of local holomorphic coordinate \( u \) and hence the metric of Eq. (4.11) may also be written as:

\[
ds^2 = \text{Im} \frac{d a_0}{d a} \frac{d a_0}{d \bar{a}} du d\bar{u} = -i \left( \frac{d a_0}{d a} \frac{d a_0}{d \bar{a}} \frac{d a}{d \bar{u}} \frac{d a}{d u} \right) du d\bar{u}
\]

which is invariant under adding a constant to \( a_0 \) or \( a \).

In Coulomb phase the gauge theory has massless photons and hence it is subjected to standard electromagnetic-duality.

\[
q = (n_e, n_m) = (-n_m, n_e)
\]

\[
q' = -\frac{1}{\tau} q
\]

where the electric charge \((1,0)\) transforms to magnetic charge \((0,1)\) which incorporates the inversion of \(\tau\). Under another duality transformation:

\[
\tau \rightarrow \tau_i + 1
\]

we have

\[
q = (n_e, n_m) \rightarrow (n_e, -n_m, n_m)
\]

which incorporates the transformation of a monopole \((0, n_m)\) to a dyon \((-n_m, n_m)\) and the transformation of a dyon \((1,1)\) to a monopole \((0,1)\). The transformations (4.13) and (4.14) generate the duality group \(SL(2,Z)\). The curve of marginal stability separating weak and strong coupling regions in moduli space is an ellipse, for the holomorphic function (4.10).

References: