Cylindrically symmetric dust distribution

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A solution for cylindrically symmetric dust distribution with magnetic and scalar field is presented here.

Banerjee\(^1\) found a solution of cylindrically symmetric distribution of matter with magnetic field. Som and Raychaudhury\(^2\) also found a solution of cylindrically symmetric charged matter distribution in rigid rotation and found that mass-density and charge density are linearly related with each other. Wolk et al.\(^3\) studied a distribution of incoherent dust with scalar field and showed that mass-density and scalar charge density are equal. In the present paper, the author has tried to find a solution of cylindrically symmetric dust distribution with repulsive scalar and magnetic field and has shown that mass-density and scalar charge-density are linearly related with each other. The solution is reduced to that of Banerjee\(^1\) when a certain parameter \(l = \frac{-2}{b+2}\) where constant \(b\) is given by \(8\pi\rho = bH^2\).

Field equation and solution — The General static cylindrically symmetric line-element is:
\[
dS^2 = g_{11} dr^2 + g_{22} r^2 d\theta^2 + g_{33} d\phi^2 = g_{ij} dr^i dr^j
\]  \(i=1, 2, 3\) and \(j=1, 2, 3\), respectively. The matter is at rest in the coordinate system of Eq. (1) so that \(\psi^0 = \delta^0_0 (g_{00})^{-1/2}\). Here magnetic field \(F^M\) exists along \(Z\) direction.

Einstein-Maxwell’s equations are:
\[
R^\mu_\nu - \frac{1}{2} R g^\mu_\nu = -8\pi T^\mu_\nu
\]  \(\mu, \nu = 0, 1, 2, 3\) \(\ldots\) \(\ldots\) (2)
\[
T^\mu_\nu = \rho V^\mu V^\nu + E^\mu E^\nu + \mathbf{B}^\mu \mathbf{B}^\nu
\]  \(\mu, \nu = 0, 1, 2, 3\) \(\ldots\) \(\ldots\) (3)

\[
E^\mu_\nu = \frac{1}{4\pi} [ - F^{\alpha\beta}_\mu F^\mu_\alpha + \frac{1}{4} \delta^\alpha_\nu F^{\alpha\beta}_\mu F^\mu_\beta ]
\]  \(\mu, \nu = 0, 1, 2, 3\) \(\ldots\) \(\ldots\) (4)
\[
4\pi \gamma K^\nu_\mu = S^i_\mu \delta^i_\nu - \frac{1}{2} \delta^\nu_\mu S^{ij} S^i_\alpha
\]  \(\mu, \nu = 0, 1, 2, 3\) \(\ldots\) \(\ldots\) (5)
\[
F^\mu_\nu = 4\pi J^\mu
\]  \(\nu = 0, 1, 2, 3\) \(\ldots\) \(\ldots\) (6)
\[
F_{[\mu, \nu, S]} = 0
\]  \(\mu, \nu, S = 0, 1, 2, 3\) \(\ldots\) \(\ldots\) (7)

where \(\rho\) is mass-density. \(S\) is the scalar field. \(\gamma\) in Eq. (5) is such that for \(\gamma = +1\), the scalar field is attractive for \(\gamma = -1\), the scalar field is repulsive. Both kinds of field satisfy the equation,
\[
S^{i\mu} = 4\pi \gamma \theta
\]  \(\mu = 0, 1, 2, 3\) \(\ldots\) \(\ldots\) (8)

where, \(\theta\) is the scalar charge-density.

Since \(R^1_1 + R^2_2 = 0\), we can use Weyl’s line-element:
\[
dS^2 = e^{2\alpha} dr^2 - e^{-2\beta} (dt^2 + dZ^2) - r^2 e^{-2\alpha} d\phi^2
\]  \(\ldots\) \(\ldots\) (9)

where, \(\alpha\) and \(\beta\) are functions of \(r\) only. Here \(t, r, \phi\) and \(\psi\) are numbered 0, 1, 2 and 3, respectively.

Eqs. (2), (6) and (8) with the help of line-element (9) give:
\[
e^{2(\alpha - \beta)} \left( \frac{\beta_1}{r} - \frac{\alpha_2}{r} \right) = H^2 + \gamma S^2 \gamma e^{2(\alpha - \beta)}
\]  \(\ldots\) \(\ldots\) (10)
\[
e^{2(\alpha - \beta)} \left( \frac{\beta_1}{r} - \frac{\beta_2}{r} \right) = -H^2 - \gamma S^2 \gamma e^{2(\alpha - \beta)}
\]  \(\ldots\) \(\ldots\) (11)
\[
e^{2(\alpha - \beta)} \left( \frac{\alpha_1^2 + \beta_1^2}{r^2} \right) = -8\pi \rho + H^2 - \gamma S^2 \gamma e^{2(\alpha - \beta)}
\]  \(\ldots\) \(\ldots\) (12)
\[
e^{2(\alpha - \beta)} \left( \frac{22 + 2 \alpha_1}{r} \right) = \frac{22 - 2 \alpha_1}{2}
\]  \(\ldots\) \(\ldots\) (13)
\[
\frac{d}{dr} \left( r S^2_1 \right) = 4\pi \gamma \theta e^{2(\alpha - \beta)}
\]  \(\ldots\) \(\ldots\) (14)
\[
\frac{d}{dr} \left( r F^2_3 \right) = 4\pi \gamma J^3 \gamma e^{2(\alpha - \beta)}
\]  \(\ldots\) \(\ldots\) (15)

where, \(F^2_3 = H^2\). Here \(J^3\) is the conduction current density along \(\phi\) direction. The suffix (1) indicates differentiation with respect to \(r\).

Note
Now, since there are five equations and seven variables let us assume,
\[ 8\pi \rho = b H^2 \]
...(16)
and, \( \alpha = l \log (1 + A r^2) \)
...(17)
where, \( b, l \) and \( A \) are constants.

Now, from Eqs (10) and (12) we have:
\[ H^2 = \frac{1}{2} \left( \frac{\beta_{11} + \beta_1}{r} \right) e^{2(\alpha - \beta)} \]
...(18)
and \( 2 \gamma S_1^2 = \left( \frac{\beta_1}{r} - \beta_{11} - 2 \alpha_i^2 \right) \)
...(19)
From Eqs (12) and (13) we have:
\[ 8\pi \rho + 2 H^2 = 2 \left( \alpha_{11} + \frac{\alpha_i}{r} \right) e^{2(\alpha - \beta)}/(2+2) \]
...(20)

using Eq. (16), we have from Eq. (20)
\[ H^2 = \left( \frac{2}{b+2} \right) \left( \alpha_{11} + \frac{\alpha_i}{r} \right) e^{2(\alpha - \beta)} \]
...(21)
Writing \( \frac{2}{b+2} = \frac{b'}{2} \)
From Eqs (18) and (21) we have:
\[ (\beta - b' \alpha)_{11} + \frac{1}{r} \left( \beta - b' \alpha_i \right) = 0 \]
...(22)
whose solution is:
\[ \beta - b' \alpha = c \]
...(23)
where \( c \) is a constant.

Thus \( \beta = b'/l \log (1 + A r^2) + c \)
...(24)
To avoid singularity, the value of \( c \) should be zero.

Hence Eqs (21) and (16) give:
\[ H^2 = \frac{2 A b' l}{\left( 1 + A r^2 \right)^{2-2l/2+2b'}} \]
...(25)
and \( 8 \pi \rho = \frac{2 A b b' l}{\left( 1 + A r^2 \right)^{2-2l/2+2b'}} \)
...(26)

Eq. (19) gives:
\[ S = \sqrt{\frac{1}{2} \left( 2 l - b' \right) \log (1 + A r^2)} \]
...(27)
From Eq. (14):
\[ \theta = \frac{1}{4 \pi} \frac{2 \epsilon}{\left( 1 + A r^2 \right)^{2/2+2l/2+2b'}} \]
...(28)
where, \( \epsilon = \sqrt{2 A b' l (b' - l)} \)

Thus, the ratio mass-density to scaler charge density:
\[ \frac{\rho}{\theta} = \frac{b b' l}{\sqrt{2 l (b' - l)}} = \text{constant} \]
...(29)

The conduction current density along \( \phi \) direction
\[ j^1 = \frac{(2 A)^{3/2} \sqrt{b' / l (1 - l)}}{4 \pi} \frac{1}{\left( 1 + A r^2 \right)^{2-3/2+2l/2}} \]
...(30)

Discussion — For mass-density and magnetic energy density to be positive, \( A, b \) and \( l \) should be positive.

For \( 1 < b' < 2l \), the mass-density, magnetic energy density and conduction current density will decrease with the increase of \( r \). The scalar field \( S \), conduction current density \( J^1 \) and constant \( e \) will be real. The mass-density and scalar charge density are found to be linearly related with each other [Eq. (29)].

For \( l = b' \), the scalar field \( S = 0 \), and hence Banerjee’s solution will be obtained.

For \( l = 1 \), \( J^1 \) will be zero and hence source-less magnetic field will be obtained.

Exterior metric — For exterior solution, \( \rho = \theta = J^1 = 0 \) and hence from Eq. (14) we have:
\[ S = f \log r \]
...(31)
where \( f \) is any arbitrary constant.
From Eqs (10) and (13) we have:
\[ (\beta - 2 \alpha)_{11} + \frac{1}{r} (\beta - 2 \alpha_i) = 0 \]
...(32)
solution of Eq. (32) is:
\[ \beta - 2 \alpha = d \]
...(33)
where \( d \) is constant
and \( 2 \alpha = m \log r \)
...(34)
where \( m \) is another constant.

Eqs (10) (12) and (31) give:
\[ \left( \frac{\beta_1}{r} - \beta_{11} - 2 \alpha_i^2 \right) = 2 \gamma f^2/r \]
...(35)
solution of (35) gives using (33),
\[ \alpha = \log \left( r^{\lambda} + h r^{2-\lambda} \right) \]
...(36)
and hence \( \beta = 2 \log \left( r^{\lambda} + h r^{2-\lambda} \right) + n \)
...(37)
where, \( \lambda = 1 \pm \sqrt{1 + f^2} \), \( h \) and \( n \) are constants.
From Eq. (35) using Eq. (34),
\[ \alpha = \log \left( r^n + K r^{2-\lambda} \right) \]
...(38)
and hence
\[ \beta = 2 \log \left( r^n + K r^{2-\lambda} \right) + (p^2 - 2 p - f^2) \log r + q \]
...(39)
where \( p, k, q \) are constants of integration.
We shall now show that the solution given in Eqs. (35) and (37) cannot be matched with the dust distribution. For this, let us consider a cylinder of matter of radius \(a\) which is matched with the magnetic and scaler field solution at the exterior. For a static metric, \(R_0^0\) and \(R_2^2\) can be expressed as:

\[
R_0^0 = r^2 = \frac{g^{11} g^{\nu \gamma} g_{\nu \gamma}}{g_{00}}
\]

...(40)

\[
R_2^2 = g^{11} g^{\nu \gamma} g_{\nu \gamma} = \frac{g^{11} g^{\nu \gamma} g_{\nu \gamma}}{g_{00}}
\]

...(41)

From the field equations we have:

\[
(R_2^2 - R_0^0) = \frac{1}{16 \pi} p \sqrt{-g}
\]

...(42)

Integrating this over unit length of radius \(a\), we have:

\[
\int_0^{2\pi} \int_0^a 16 \pi \rho \sqrt{-g} \, dr \, dz \, d\varphi
\]

...(43)

Substituting the values from Eqs (36) and (37) we have:

\[
\text{mass} = \int_0^{2\pi} \int_0^a \rho \sqrt{-g} \, dr \, dz \, d\varphi
\]

\[
= \frac{1}{4} (P - 1 + \sqrt{1 + f^2}) (1 - P + \sqrt{1 + f^2})
\]

...(44)

The mass inside the cylinder will be zero if,

\[
P = 1 \pm \sqrt{1 + f^2}
\]

For this value of \(P\), the solution given by Eqs (33) and (39) will be reduced to the solution given by Eqs. 36 and (37). The solutions given in Eqs. (36) and (37) thus correspond to mass-less space.

This solution cannot be matched with the dust distribution.

**Continuity with the exterior solution** — For continuity at \(r = r_b\) we have:

\[
(1 + A r_b^p)^{1/2} = r_b^{p} + K r_b^{-2 - p}
\]

...(45)

\[
2 A l r_b^{p - 1} = \frac{P r_b^{p - 1} + k (2 - P) r_b^{1 - p}}{1 + A r_b^{p - 1} + k r_b^{-2 - p}}
\]

...(46)

\[
(1 + A r_b) = (r_b^{p} + K r_b^{-2 - p}) r_b^{-2 - p}
\]

...(47)

\[
2 A l b^{p - 1} r_b^{p - 1} + 2 k (2 - P) r_b^{1 - p}
\]

\[
\frac{1 + A r_b^{p - 1} + k r_b^{-2 - p}}{r_b^{p} + k r_b^{-2 - p}}
\]

...(48)

Thus the four unknown \(A, P, K\) and \(f\) may be determined by the four equations.

**References**