Vacuum structure, stability and non-triviality of $\lambda \phi^4$ theory

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Through an explicit construct of the vacuum state in $\lambda \phi^4$ theory by Bogoliubov transformation the authors show variationally that the theory can be rendered stable and non-trivial. The structure of the vacuum so constructed is characterised by a condensate of off-shell particle pairs. The effective potential based on the vacuum emerges identical to that obtained in the Gaussian approximation. In contrast, the effective potential based upon the perturbative vacuum is shown to lead either to an unstable theory or to a trivial one. The two-point correlation-function and the momentum distribution function for the particle-condensate in the vacuum (vacuum structure function) are calculated.

I Introduction

Understanding the structure of the vacuum state in quantum field theories is a basic problem both from theoretical and the phenomenological point of view. It becomes particularly important when one is dealing with a strongly interacting system such as described by quantum chromodynamics (QCD). It is, for example, believed that the QCD-vacuum is endowed with a highly non-trivial structure characterised by gluon- and quark-condensates which are of purely non-perturbative origin. Although attempts have been made to dynamically understand the structure of the QCD-vacuum, these require rather non-trivial simplifying assumptions owing to the intractability of the non-perturbative sector of the theory. It may, therefore, be instructive to glean some insight into the structure of the vacuum in some simpler field-theories such as the $\lambda \phi^4$ theory which is free from the complications due to the presence of fermions and the non-Abelian nature of QCD. The hope is that the quartic self-coupling in the $\lambda \phi^4$ theory may simulate the essential non-perturbative features due to the analogous quartic self-interactions of the gluons in QCD.

Further, it may be emphasised that the $\lambda \phi^4$ theory, quite independently on its own, has become the subject of intense investigation in the recent literature presumably because it provides a laboratory for testing important theoretical ideas including: study of convergence and behaviour at large orders of perturbation theory, nature of phase transition and critical phenomena, the Higgs-Kibble mechanism and spontaneous symmetry breaking, effective theories of superconductivity (e.g. the Ginzburg–Landau model), superfluidity and inflationary cosmology etc.

However, in spite of the inherent simplicity, the $\lambda \phi^4$ theory is far from being adequately understood at present. In particular, the important question of non-triviality of this theory is being fiercely debated. While studies based upon lattice calculations indicate the triviality-scenario, several analytic-investigations (mostly based upon continuum-variational calculations) have demonstrated otherwise.

In this paper, the authors attempt to shed some light on the structure of the vacuum as well as the triviality issue in $\lambda \phi^4$ theory. In particular, they seek to investigate possible connections, if any, that may exist between the above two aspects of the theory. The paper is organised as follows: In Section 2, they construct the trial-vacuum state (hereinafter referred to as the interacting vacuum state (IVS)) employing the technique of the Bogoliubov transformation. The requirement of variational minimisation of the energy then leads to a gap-equation for the characteristic function which governs the structure
of the IVS. The effective potential (EP) based upon the IVS is obtained and the programme of renormalisation is implemented in Section 3. In the same section the authors also demonstrate the equivalence of the results with those of the Gaussian approximation and establish the non-triviality of the ensuing renormalised theory. In the next Section 4, the stability of the trial-vacuum state is established with respect to the free-field (perturbative vacuum) by comparing the respective EP's. In this Section it is also demonstrated that the perturbative vacuum leads either to an unstable-or a trivial theory. In Section 5, they explore further implication of the IVS by establishing the non-trivial structure of the former characterised by a condensate of off-shell, correlated particle-pairs. The authors also calculate the two particle correlation-function and interpret the result in terms of an inter-particle potential. In Section 6, they conclude with a summary of the main results and a comparison with parallel investigations.

2 Variational Ansatz for the Interacting Vacuum and the Resulting Gap-Equation

We consider the theory in the massive, symmetric-phase described by the Lagrangian:

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi \right) - \frac{1}{2} m^2 \phi^2 - \lambda \phi^4$$

where \( m > 0 \). The Hamiltonian density following from the above Lagrangian is given by:

$$\mathcal{H} = \frac{1}{2} \left( m^2 \phi^2 + \frac{1}{4} \phi_x^2 + \frac{1}{4} \phi_y^2 + \frac{1}{4} \phi_z^2 \right) + \frac{1}{2} \lambda \phi^4$$

where \( \phi_x = \partial \phi / \partial x, \phi_y = \partial \phi / \partial y, \phi_z = \partial \phi / \partial z \).

The field \( \phi \) can be Fourier-expanded in terms of the free-field operators in the standard fashion:

$$\phi(x,t) = \sigma + \frac{1}{\Omega(k)} \left[a(k)e^{ikx} + a^\dagger(k)e^{-ikx}\right]$$

where, \( k \neq k^o-t-k_x \sigma \) is a real, constant background field equal to the vacuum expectation value (VEV) of \( \phi \) and is required to define the EP:

$$\Omega(k) = 2\omega(k)(2\pi)^3$$

and \( \omega(k) = k^0 = (k^2 + m^2)^{1/2} \).

The free-field operators obey the standard commutation relation:

$$[a(k), a^\dagger(q)] = \Omega(m) \delta^3(k - q)$$

and the free-field vacuum is defined by:

$$a(k)|\text{vac}> = 0, \text{for arbitrary } k$$

However, in the presence of \( \lambda \phi^4 \) self-interaction the structure of the vacuum is expected to be considerably altered. The altered structure of the vacuum/ground state in presence of interactions have been demonstrated in the case of superconductivity\(^{12}\), superfluidity\(^{11}\), the hard-sphere Bose-gas\(^{17}\) etc. This may, in all probability be a general result\(^{18}\). It is therefore, proposed the following ansatz for the trial vacuum-state denoted as \( |\text{vac}> \) :

$$|\text{vac}> = \exp \left[ -\frac{1}{2} \int \frac{d^3k}{\Omega(m)} \beta(k) a(k)a(-k) \right] |\text{vac}>$$

where \( \beta(k) \) is an arbitrary, real function of \( k \) i.e. \( \beta(-k) = \beta(k) \) which characterises the departure from the free-field (perturbative) vacuum with the limiting behaviour:

$$|\text{vac}> \rightarrow |0> \text{ as } \beta(k) \rightarrow 0.$$ The motivation for the above ansatz comes primarily from the successful description\(^6\) of the physics of the anharmonic-and the double-well oscillator with analogous structure of the ground state and from other variational studies in model field theories\(^{19}\).

A set of (interacting) field operators \( b(k), b^\dagger(k) \) can be constructed by the technique of Bogoliubov transformation\(^6\) which satisfy the same canonical commutation relations and are associated with the trial vacuum state, i.e.:

$$[b(k), b^\dagger(q)] = \Omega(m) \delta^3(k - q) \text{ and } b(k)|\text{vac}> = 0$$

The required transformation is given by:

$$b(k) = \cosh(\alpha(k)) a(k) - \sinh(\alpha(k)) a(k) \text{ and } b^\dagger(k)$$

$$b^\dagger(k) = \cosh(\alpha(k)) a^\dagger(k) - \sinh(\alpha(k)) a^\dagger(k)$$

where

$$\alpha(k) = \tanh^{-1}(\beta(k))$$

It is useful to have the inverse transformation which is given below:

$$a(k) = \cosh(\alpha(k)) b(k) + \sinh(\alpha(k)) b^\dagger(-k)$$

$$a^\dagger(k) = \cosh(\alpha(k)) b^\dagger(k) + \sinh(\alpha(k)) b(-k)$$
The effective potential can be obtained by calculating the VEV of the Hamiltonian. Use of translational invariance and the inverse transformations Eq. (8), facilitates the calculation and we obtain:

\[
\langle \mathcal{H}(x,t) \rangle = \frac{1}{2} m^2 \sigma^2 + \frac{1}{2} \lambda \sigma^4 + \int \frac{d^4 k}{\Omega_k(m)} \left[ k^2 u(k) + \omega_k^2 / u(k) \right]
\]

(9)

where

\[
\langle \mathcal{H}(x,t) \rangle = \langle \text{vac} \mid \mathcal{H}(x,t) \mid \text{vac} \rangle = \langle \text{vac} \rangle \langle \text{vac} \rangle \quad \ldots \quad (10)
\]

\[
u(k) = \exp(2\alpha(k)) \quad \ldots \quad (11)
\]

and

\[
\bar{T}_0(m) = \int \frac{d^4 k}{\Omega_k(m)} u(k) \quad \ldots \quad (12)
\]

The arbitrary function \(u(k)\) is variationally fixed by functional-minimisation of \(\langle \mathcal{H} \rangle\) and is given by:

\[
u(k) = \omega_k(m)/\omega_k(M) = [k^2 + m^2] / (k^2 + M^2)^{1/2} \quad \ldots \quad (13)
\]

where

\[
M' = m^2 + 12 \lambda \sigma^2 + 12 \lambda \bar{T}_0(m) \quad \ldots \quad (14)
\]

Note that Eq. (14) is a self-consistency condition (hereinafter referred to as the gap-equation) for the parameter \(M'\). This is made explicit by rewriting Eq. (14) as:

\[
M' \equiv m^2 + 12 \lambda \sigma^2 + 6 \lambda \int \frac{d^4 k}{(2\pi)^3} (k^2 + M^2)^{1/2} \quad \ldots \quad (15)
\]

Note that:

\[
M' > 0 \quad \ldots \quad (16)
\]

by virtue of Eq. (14) and the reality of \(u(k)\). (The case: \(M' = 0\), is ruled out as it would imply, by Eq. (14), a relation between \(\lambda, m\) and \(\sigma^2\) which are independent parameters in the theory, by definition.) It is now straight-forward to determine the EP (next Section).

### 3 Effective Potential and Renormalisation

The effective potential \(V(\sigma)\) is defined as the minimum of \(\langle \mathcal{H} \rangle\) with respect to functional variation of the arbitrary function \(u(k)\). Therefore, it is obtained by substituting Eqs (11 and 12) in Eq. (9). This yields:

\[
V(\sigma) = \frac{1}{2} m^2 \sigma^2 + \frac{1}{2} \lambda \sigma^4 + I_0(M) - 3 \lambda \bar{T}_0^2(M) \quad \ldots \quad (17)
\]

where

\[
I_0(M) = \frac{1}{2} \int \frac{d^4 k}{(2\pi)^3} (k^2 + M^2)^{1/2} \quad \ldots \quad (18)
\]

\[
I_f(M) = \frac{1}{2} \int \frac{d^4 k}{(2\pi)^3} (k^2 + M^2)^{1/2} \quad \ldots \quad (19)
\]

eetc. In general, these integrals can be defined as

\[
I_n(\chi) = \int \frac{d^4 k}{\Omega_k(\chi)} \left[ \phi_k(x) \right]^n, \quad n = 0, \pm 1, \pm 2, \ldots \quad (20)
\]

The authors at once recognised that the gap-equation, Eq. (14) and the integrals defined by Eqs (18, 19) are identical to those obtained in the Gaussian-approximation\(^{11}\). The rederivation of the results of the Gaussian-effective potential\(^{11}\) (GEP) by the present method starting from the vacuum-structure ansatz, Eq. (4) is considered rather significant as it clearly links the dynamical origin of the GEP to the altered vacuum structure in presence of interaction.

In order to extract the physical content of the theory in this variational approach, it is necessary to implement the renormalisation programme. This task is, however, facilitated by the above identification with the results of the GEP where such a programme has already been carried out. Therefore, it is sufficient for our purpose to present here the main results.

The renormalised parameters, denoted by \(\lambda_{R0}\) and \(m_{R0}\) are defined in the standard manner by first fixing the vacuum-configuration which corresponds to the global minimum of the effective potential in \(\sigma\), and then relating these parameters to the derivatives of the EP at the minimum:

\[
m_{R0}^2 = d^2 V(\sigma) / d\sigma^2 \bigg|_{\sigma=\sigma_0} \quad \ldots \quad (21)
\]

\[
\lambda_{R0} = (1/4!) d^4 V(\sigma) / d\sigma^4 \bigg|_{\sigma=\sigma_0} \quad \ldots \quad (22)
\]
For the case considered here (i.e. \( m > 0 \)), the minimum of the EP occurs at \( \sigma = 0 \). From Eqs (17), (21) and (22) we obtain, with the help of the gap-Eq. (14), the following results:
\[
m_R^2 = m^2 + 12 \lambda I_6(m_R) = M^2 (\sigma = 0) \quad (23)
\]
\[
A = \lambda [ 1 - 12 \lambda I_4(m_R) ] [ 1 + 6 \lambda I_6(m_R) ]^{-1} \quad (24)
\]
where the integrals, \( I \), are defined by Eq. (20).

An important consequence of Eq. (23) is that \( \lambda \) must be chosen negative: \( \lambda < 0 \), in order that the theory becomes renormalisable. This is because \( I^0(m_R) \) is divergent and the (unmeasurable) bare-mass \( m \) of the theory may be infinite but the physical (renormalised) mass \( m_R \) has to be finite. The Eqs (23-24) as well as the above conclusions are also true in the Gaussian approximation\(^{11}\).

In order to investigate the stability and non-triviality of the theory it is necessary to invert Eq. (24). This yields two solutions for the bare-coupling \( \lambda \) of which, the physically acceptable solution is given by the following:
\[
\lambda = [ -1/6 I_4(m_R) ] [ x [ 1 + 2( \lambda I_6(m_R) ) ] + ... ] \quad (25)
\]
where the neglected terms are sub-leading. (The other solution for \( \lambda \) is given by:
\[
\lambda = (1/2) A R + 0(1/I_4(m_R)) \].)

This solution can be shown to lead to an unstable minimum of the EP in this case, lies (infinitely) higher than that corresponding to Eq. (25). Substituting this solution for the bare-coupling, the corresponding solution for the bare-mass is obtained:
\[
m^2 = m_R^2 + 2 A(m_R)/I_4(m_R) + (\text{sub-leading terms}) \quad (26)
\]

It is important to recall at this point the basic principle of renormalisation, i.e., the physical manifestation of the theory is through the renormalised parameters which are observables and which must also lead to an acceptable theory requiring, e.g. finiteness, stability and uniqueness. The unobservable bare-parameters appearing in the defining Lagrangian must, therefore, be chosen in accordance to the above requirement, as has been followed here.

Next, it is straightforward to recast the EP, given by Eq. (17), in manifestly renormalised form by the help of Eqs (25-26) and by the use of the subtraction procedure employing the Stevenson identities\(^{11}\). One then obtains after some algebra, the following expression for the EP:
\[
V(\sigma) = V_{\text{max}} + \frac{1}{4} m_R^2 \sigma^2 - \frac{m_R^4}{128 \pi^2} (t - 1)^2 - \frac{m_R^2}{64 \pi^2} (t - 1) \eta \quad (27)
\]

The renormalised version of the gap-equation is similarly obtained:
\[
(1 - \eta) (t - 1) - (16 \pi^2/ m_R^2) \sigma^2 = t \ln t \quad (28)
\]

In the above Eq. (28), the following notation is used:
\[
\sigma = M^2(\sigma)/m^2, \eta = -4 \pi^2 \beta_R.
\]

and
\[
V_{\text{max}} = I_1(m_R) - 3 \lambda I_6(m_R) \quad (29)
\]

It must be emphasised at this point, that \( t = t(\sigma) \) of the gap Eq. (28) must be obtained first which is then to be substituted in Eq. (27) to infer the functional dependence of \( V(\sigma) \) on \( \sigma \). Note that the solution of the transcendental Eq. (28) for \( \sigma^2 \) exists only when the latter is less than some upper-limit, given by:
\[
\sigma^2 \leq (m_R^2/16 \pi^2) [ e^{-\eta} + \eta - 1 ] = \sigma^2_{\text{max}} \quad (30)
\]

The domain of validity of the EP is thus determined by the Eq. (30). In particular, in the large coupling regime \((\eta \rightarrow 0)\), the domain of the EP shrinks with \( \eta \) since \( \sigma_{\text{max}} \rightarrow 0 \) in this limit. This is the case of small oscillations about the minimum located at \( \sigma = 0 \) and corresponds to the pathological situation when \( | \lambda_R | \rightarrow \infty \). On the other hand, in the small coupling regime, which corresponds to the limit \( \eta >> 1 \), the domain of \( V(\sigma) \) increases with \( \eta \). It is reasonable, therefore, to expect that the true effective potential exhibits similar behaviour unless the variational approximation and the vacuum-structure ansatz grossly misrepresent the actual situation.

In summary, the EP is given by Eq. (27) and the resultant renormalised parameters (as given by Eqs (23) and (24)) render the theory stable and non-trivial, the physical content being fully specified by
the interacting vacuum state, Eq. (4) and the associated field operators: \( b(k) \) and \( b^\dagger(k) \). On the contrary, if the theory is developed about the free-field (perturbative) vacuum, then one is led either to instability or to a trivial theory. This is demonstrated in the following section.

4 Stability of the Perturbative Vacuum and Triviality

The effective potential \( V^\prime(\sigma) \) based on the perturbative vacuum \( |0> \) can be simply inferred from Eq. (9) by setting \( \alpha(k) = 0 \) \( (u(k) = 1) \). This leads to the following:

\[
V^\prime(\sigma) = \frac{1}{2}m^2\sigma^2 + \lambda\sigma^4 + 6\lambda\sigma^2I_0 + 3\lambda'I^2 + I_1, \tag{31}
\]

where \( I_n \equiv I_n(m) \) are the integrals as defined in Eq. (20). The renormalised parameters (denoted by \( \bar{m}_R \) and \( \bar{\lambda}_R \)) derived from \( V^\prime(\sigma) \) in a likewise manner, are given by:

\[
\bar{m}_R = m^2 + 12\bar{\lambda}I_0 \tag{32}
\]

\[
\bar{\lambda}_R = \lambda \tag{33}
\]

(The difference between the above Eqs (32-33) and the corresponding Eqs (23-24) may be carefully noted as it is crucial for the analysis to follow. The difference arises because the integrals, \( I_n \) are independent of \( \sigma \).

Using these renormalised parameters, \( V^\prime(\sigma) \) can be recast in the renormalised form:

\[
V^\prime(\sigma) = \frac{1}{2}\bar{m}_R\sigma^2 + \bar{\lambda}_R\sigma^4 + 3\bar{\lambda}_R'I^2 + I_1 \tag{34}
\]

If one constrains the renormalised parameters, \( \bar{m}_R \) and \( \bar{\lambda}_R \) to be finite as required of any renormalisable theory then Eqs (32) and (33) imply that \( \bar{\lambda}_R = \lambda \leq 0 \). However, the case: \( \bar{\lambda}_R < 0 \) is ruled out on grounds of stability since this would lead to the non-existence of a lower-bound of \( V^\prime(\sigma) \) as can be seen from Eq. (34). Thus, to avoid instability, the only other option is: \( \bar{\lambda}_R = 0 \), which further implies: This is nothing but the triviality scenario of the \( \bar{m}_R = m \) theory!

It is believed that the above constitutes a straightforward demonstration that in the implementation of the physical consequences of the theory based upon the free-field (perturbative) vacuum, triviality emerges as an inescapable conclusion! Further evidence for triviality of the theory (when developed about the perturbative vacuum) comes from the calculation of the two-point correlation function.

5 Properties of the Interacting Vacuum State

The vacuum-structure ansatz, Eq. (4) has the important consequence that the interacting vacuum state: \( |\text{vac}> \) is not devoid of particle-content in contrast to the case of the perturbative-vacuum: \( |0> \). To demonstrate this, the authors calculate the number-density: \( u(k) \) of the free-field quanta residing in \( |\text{vac}> \). Noting that the (free-field) number-operator is given by the standard expression: \( a(k) \), the desired number-density is given by the expression \( (v = \text{spatial volume})\):

\[
n(k) = \frac{1}{v}a(k) a(k) \tag{35}
\]

Using Eq. (8), \( n(k) \) is easily computed and given by:

\[
n(k) = \sinh^2 \left[ \frac{\alpha(k)}{2\pi} \right] / (2\pi)^3 \tag{36}
\]

where \( \alpha(k) \) can be obtained from Eqs (11) and (13). This leads to the following result for the number-density:

\[
n(k) = (1/32\pi^2)(|\omega_k(m)/\omega_k(m_R)| \tag{37}
\]

Note that the bare-mass \( m \) is divergent \((m/m_R) \sim O / \sqrt{\ln\Lambda}\) (see Eq. (26)) where \( \Lambda \) is the momentum-cutoff. Since, according to the standard procedure of renormalisation, the cut-off must be removed (i.e., sent to infinity) prior to calculation of any physical quantity in the theory, one obtains:

\[
\lim n(k)/n(0) = \rho(k) = [1 + (k^2/m_R^2)]^{1/2} \tag{38}
\]

(\( \Lambda \to \infty \))
where \( n(0) = (m/m_r)/(2\pi^2) \) is the maximum value of \( n(k) \).

Eq. (38) indicates condensation of off-shell particle-pairs with correlated momenta (of equal magnitude but opposite direction). This feature is not surprising as it is built into the vacuum-structure ansatz, Eq. (4). What is, perhaps, non-trivial about the above result, Eq. (38), is the momentum-dependence of \( \rho(k) \). It shows that there is a significant fraction of the particle-pairs with non-zero value of \( |k| \). This is expected to have important observable consequences, particularly at finite temperature.

They next calculate the two-point correlation function \( U(x-y) \) in the interacting vacuum, defined by:

\[
U(x-y) = \langle \phi(x,0) \phi(y,0) \rangle_{\text{vac}} - \langle \phi(x,0) \rangle_{\text{vac}} \langle \phi(y,0) \rangle_{\text{vac}} \quad \ldots (39)
\]

As is well known, this function carries important information regarding the inter-particle potential in the medium in which it propagates (the interacting vacuum in the present case). Using translational invariance and Eq. (8), we obtain:

\[
U(x-y) = \int \frac{dk}{\Omega(k)} e^{i(k,x-y)} + e^{-i(k,x-y)} \quad \ldots (40)
\]

From Eqs (11) and (13) we infer that:

\[
e^{i(k,x-y)} = \omega_k(m) / \omega_k(m_k).
\]

Substitution of this result in Eq. (40) above leads, finally to the following expression for \( U(r) \):

\[
U(r) = \frac{1}{2} \int \frac{dk}{(2\pi)} e^{ikr} (k^2 + m_k^2)^{-1/2} \quad \ldots (41)
\]

where \( r = x - y \). The above integral can be related to the modified Bessel function \( K(m_k r) \):

\[
U(r) = m_k K(m_k r) / 4\pi^2 r \quad \ldots (42)
\]

where \( r = ||r|\). Note that \( U(r) \) diverges in the limit, \( r \to 0 \) as it should (in any local quantum field-theory) and behaves asymptotically as:

\[
\lim_{r \to \infty} U(r) \sim e^{-m_k r} r^{-1/2} \quad \ldots (43)
\]

which shows a fall-off faster than the Yukawa-potential! To our knowledge, the important information (contained in Eqs (41–43) regarding the static–potential of the symmetric \( \lambda \phi^4 \) theory has been derived here for the first time.

It is instructive to repeat the above calculation for the case of the perturbative vacuum, \( 10 > \) for comparison. Denoting by \( U^p(r) \) the analogous correlation function based upon the perturbative vacuum, we obtain the former by simply setting \( \alpha_n^{(k)} \) in Eq. (40). This leads to:

\[
U^p(r) = \frac{1}{2} \int \frac{dk}{(2\pi)} e^{ikr} (k^2 + m)^{-1/2} = m K_0(m r) / 4\pi^2 r \quad \ldots (44)
\]

However, since the bare-mass \( m \) is divergent (see, Eq. (26)) as the cut-off \( \Lambda \) is first sent to \( \infty \) (see, Eq. (26)) one obtains the trivial result that \( U^p(r) \) vanishes for all \( r \) which is the characteristic behaviour of free-field theory! This is yet another demonstration of the triviality of the theory when perturbatively implemented.

6 Summary and Conclusions

The main results of this work is the demonstration, by variational methods, that the physical properties of \( \phi^4 \) field theory depend critically on the structure of the physical vacuum state. It is shown that the assumed ansatz, Eq. (4) for the interacting vacuum state leads to a non-trivial and stable theory with finite renormalised parameters whereas, the physics developed around the non-interacting (‘perturbative’) vacuum becomes either unstable or trivial.

It is well-known, however, that lattice investigations\(^{11–15} \) of \( \phi^4 \)-field theory indicate the triviality scenario and miss the non-trivial version arrived at here and in Ref. (14). This result can be succinctly understood as follows: the lattice-regularised version of the theory corresponds to a finite ultraviolet-cut-off. This means that the bare coupling \( \lambda \) is small – 0 (1/\( L^4 \)) and negative for the case considered in this work. However, the range of the classical field \( \sigma \) remains unrestricted. For this reason, there always exist sufficiently large values of \( \sigma \) (for any given lattice-spacing) such that the term \( \lambda \sigma^4 \) occurring in the effective potential (see, Eq. (17)) dominates over all other terms (Note that the integrals, \( I_n \) are all finite on the lattice).
Therefore, $V_{\text{finite}}(\sigma)$ becomes unbounded from below since $\lambda$ is negative.

In a continuum theory, however, the ultra-violet cut-off is never actually present: if a cut-off is introduced to regularise the theory, the same has to be sent to infinity first prior to considering any other limiting behaviour, such as $|\sigma| \to \infty$. This crucial difference in the order of taking limits: (UV-Cut-off) $\Lambda \to \infty$ and $|\sigma| \to \infty$, makes all the difference in the physical content of the theory in the two approaches and explains why, for any finite lattice-spacing, it will not be possible to discover the stable and non-trivial version of the theory presented in this work and in Ref. (14). For a detailed discussion on this important point, see Ref. (14).

The present approach to $\lambda \phi^4$ theory reproduces the main results of the Gaussian approximation\(^7\). This is considered quite significant since the two approaches are based upon rather different physical assumptions. The authors consider the present approach to be more general than the Gaussian approximation since the former provides a dynamical explanation of the latter through the mechanism of altered vacuum-structure induced by the interaction. Besides, the authors go beyond the scope of the Gaussian-approach in establishing new results, e.g. the calculation of the momentum-distribution of the condensate-structure function ‘n(k)’ and the two-point vacuum correlation function $U(x-y)$ related to the inter-particle static-potential, $V(r)$.

The resulting momentum distribution of the condensate-structure function $n(k)$ deserves special mention as it displays the non-standard feature of an appreciable spread in $k$ about the origin scaled by the renormalised mass of the physical quanta. It is reasonable to expect that this condensate-structure of the physical vacuum persists to finite temperature manifesting in observable consequences in the thermodynamic properties of the associated system. This would, therefore, constitute a test of the basic underlying assumption of the vacuum-structure. Similarly, the behaviour of the derived inter-particle potential showing a fall-off faster than the Yukawa potential is a direct prediction having interesting testable consequences.

Finally, the authors comment on related work in the recent literature. As has been remarked earlier, analogous ansatz for the ground state and the field--operators derived by Bogoliubov transformation, has been used in Ref. (6) for the case of the anharmonic oscillator. The important conclusion that emerges from this study\(^7\) is that a convergent and accurate perturbation theory for the energy levels results when the theory is developed about the trial vacuum-state. In contrast, the perturbation theory is badly divergent\(^7\) if developed about the non-interacting (perturbative) vacuum.

The relation of the present variational approximation (which is equivalent to the Gaussian approximation) to the one-loop approximation method\(^7\) has been discussed in detail in Ref. (14). It has been shown that the one-loop results are obtained, as a special case, in the present results (Eqs (23-24)) in the limit of small bare-coupling, $\lambda \to 0$, when $0(\lambda^2)$ terms are neglected. This means that the one-loop results remain essentially perturbative in nature, even though one resums an infinity of ordinary Feynman diagrams at the one-loop order. Besides, the one-loop effective potential has a rather restricted domain in both $\lambda$ and $\sigma$ beyond which it shows pathological behaviour. In contrast, the effective potential based upon the present variational approximation has a considerably larger range of validity and is genuinely non-perturbative in nature.

In a spirit similar to the present approach, massless $\lambda \phi^4$ theory has been variationally investigated in Ref. (24). It has been shown in this work that the preferred vacuum is also described by a condensate structure albeit with a different momentum distribution function of the condensate--particle density. In lower (1+1) dimension, the GEP has been derived\(^7\) for the $\lambda \phi^4$ theory employing similar ansatz for the trial vacuum state. However, the underlying condensate structure, the correlation function and renormalisation have not been investigated in these works.

As remarked earlier, possible applications of the results derived here are envisaged in diverse area of current interest including finite temperature field theory\(^7\), critical phenomena (involving a scalar field as the order--parameter\(^7\)), inflationary cosmology\(^7\), exploration of the vacuum structure\(^7\) of pure
gluonic-QCD and Higgs sector of the standard model (by extending) the analysis to the spontaneously broken phase, which correspond to the case of negative bare-mass $m^2 < 0$.

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