Order reduction using the advantages of differentiation method and factor division algorithm

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The authors present a combined method for order reduction of linear dynamic high order systems using the advantages of the differentiation method and factor division algorithm. The denominator of the reduced order model is obtained by the differentiation method and the numerator coefficients of the reduced order model are determined by the factor division algorithm. The reduction procedure is simple and efficient. The proposed algorithm has been extended for the order reduction of linear stable multivariable system. Two numerical examples are solved to illustrate the proposed combined method and integral square error is calculated between original and reduced order models using MATLAB software.

Keywords: Model, Order reduction, Differentiation Method, Factor Division Algorithm, Stability

In many practical situations, one can obtain a fairly complex and high order model for a system from analytic considerations. For example, consider a multi-machine power system. The model for each synchronous generator including prime-mover, the governor and the exciter can be obtained in this manner, but the order of the model turns out to be fairly high. This complexity often makes it difficult to obtain a good understanding of the behaviour of the system. The preliminary design and optimization of such systems can often be accomplished with greater ease if a low-order linear model is derived which provides a good approximation to the system.

A wide variety of model order reduction methods have been proposed by the several researchers both in time domain and frequency domain. Some text books and research monographs exclusively devoted in model reduction technique. Model order reduction of single/multi variable systems using differentiation technique has been proposed by some researchers. The factor division algorithm introduced by Lucas in which dominant poles are retained and initial time moments are preserved. This method was extended by Lucas to generate biased reduced order models by retaining initial markov parameters as well as time moments.

A combined method using the advantages of the differentiation method and factor division algorithm is proposed for single as well as multivariable linear dynamic systems. In this method the reduced denominator is obtained by the differentiation method and numerator of the reduced model is determined by the factor division. The proposed method is illustrated with the help of two examples and the step responses of reduced models are compared with the step response of original system using their plots and integral square error indices using MATLAB 7.0.

Description of Algorithm

Let the transfer function of the original high order linear dynamic SISO system of the order \( n \) be

\[
G(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{n-1} a_i s^i}{\sum_{i=0}^{n} b_i s^i} \quad \ldots (1)
\]

where \( a_i \) and \( b_i \) are known scalar coefficients.

and let the corresponding \( k^{th} \) order reduced model is synthesized as

\[
g_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{\sum_{i=0}^{k-1} c_i s^i}{\sum_{i=0}^{k} d_i s^i} \quad \ldots (2)
\]

where \( c_i \) and \( d_i \) are unknown scalar coefficients.

Further, the method consists of the following two steps.

Step-1: Determination of the denominator polynomial of the reduced order model
For getting the denominator of the \( k \)th order reduced model from the original system Eq. (1), first the denominator polynomial \( D(s) \) is reciprocated as

\[
D(s) = \frac{1}{s} \left[ \frac{1}{s} \right] \quad \ldots (3)
\]

then above polynomial Eq. (3) is differentiated \((n-k)\) times and resulting polynomial reciprocated back gives

\[
D_k(s) = \sum_{i=0}^{k} d_i s^i = d_0 + d_1 s + d_2 s^2 + \ldots + d_k s^k \quad \ldots (4)
\]

For convenience \( D_k(s) \) is written in the normalized form such that \( d_k=1 \).

**Step-2:** Determination of the numerator coefficients of reduced order models

After obtaining the reduced denominator the numerator of reduced model is determined by factor division algorithm\(^\text{13}\).

\[
N(s)D_k(s) = (a_0 + a_1 s + a_2 s^2 + \ldots + a_{n-1}s^{n-1}) \times (d_0 + d_1 s + d_2 s^2 + \ldots + d_k s^k)
\]

\[
= p_0 + p_1 s + p_2 s^2 + \ldots + p_{n-1}s^{n-1} + \ldots + p_{n+k-1}s^{n+k-1}
\]

\[
\ldots (5)
\]

and

\[
D(s) = b_0 + b_1 s + b_2 s^2 + \ldots + b_n s^n
\]

\[
\alpha_0 = \frac{p_0}{b_0}, \quad p_1, \ldots, p_{k-1}
\]

\[
\alpha_i = \frac{q_0}{b_0}, q_1, \ldots, q_{k-2}
\]

\[
\alpha_2 = \frac{r_0}{b_0}, r_1, \ldots, r_{k-3}
\]

\[
\alpha_{k-2} = \frac{u_0}{b_0}, u_1
\]

\[
\alpha_{k-1} = \frac{v_0}{b_0}
\]

where

\[
a_i = p_{i+1} - \alpha_i b_i, \quad i = 0, 1, \ldots, k - 2
\]

\[
r_i = q_{i+1} - \alpha_i b_i, \quad i = 0, 1, \ldots, k - 3
\]

\[
v_0 = u_1 - \alpha_{k-2} b_1
\]

now finally the \( k \)th order reduced model can be written as

\[
G_k(s) = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_{n-1}s^{n-1}}{D_k(s)} \quad \ldots (8)
\]

**Extension to Multivariable System**

Let the transfer function matrix of the high order \('n'\) having \('p'\) inputs and \('m'\) outputs be

\[
\begin{bmatrix}
    a_{11}(s) & a_{12}(s) & \cdots & a_{1p}(s) \\
    a_{21}(s) & a_{22}(s) & \cdots & a_{2p}(s) \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1}(s) & a_{m2}(s) & \cdots & a_{mp}(s)
\end{bmatrix}
\]

\[
\ldots (9)
\]

or \([G(s)] = [g_{ij}(s)], \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, p\) is a \(m \times p\) transfer matrix.

The general form of \( g_{ij}(s) \) of \([G(s)]\) is taken as

\[
g_{ij}(s) = \frac{a_{ij}(s)}{D(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_{n-1}s^{n-1}}{b_0 + b_1 s + b_2 s^2 + \ldots + b_n s^n}
\]

\[
\ldots (10)
\]

Let the transfer function matrix of the lower order \('k'\) having same numbers of inputs and outputs to be synthesized as

\[
\begin{bmatrix}
    b_{11}(s) & b_{12}(s) & \cdots & b_{1p}(s) \\
    b_{21}(s) & b_{22}(s) & \cdots & b_{2p}(s) \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{m1}(s) & b_{m2}(s) & \cdots & b_{mp}(s)
\end{bmatrix}
\]

\[
\ldots (11)
\]

or \([G_k(s)] = [r_{ij}(s)], \quad i = 1, 2, \ldots, m \quad \text{and} \quad j = 1, 2, \ldots, p\) is a \(m \times p\) transfer matrix.

The general form of \( r_{ij}(s) \) of \([G_k(s)]\) can be written as

\[
r_{ij}(s) = \frac{b_{ij}(s)}{D(s)} = \frac{e_0 + e_1 s + e_2 s^2 + \ldots + e_{k-1}s^{k-1}}{f_0 + f_1 s + f_2 s^2 + \ldots + f_k s^k}
\]

\[
\ldots (12)
\]
for getting $k^{th}$ order reduced transfer matrix $[G_k(s)]$, first of all common denominator $D(s)$ of Eq. (9) is reciprocated, differentiated $(n-k)$ times and again reciprocated back. Then using factor division algorithm the coefficients of numerators of each element of transfer matrix are successively determined.

**Method for Comparison**

In order to check the goodness of the reduced order models, the integral square error index ISE\textsuperscript{15} is calculated between transient parts of the original system and reduced order models. The smaller ISE indicates that the reduced order models are closer to the original system.

$$ISE = \int_0^\infty (y(t) - y_r(t))^2 dt$$

where $y(t)$ and $y_r(t)$ are the unit step responses of the original system and reduced order model.

**Numerical Examples**

**Example 1:** Consider a fourth order system described by the transfer function $G_4(s)$.

$$G_4(s) = \frac{2400 + 1800s + 496s^2 + 28s^3}{240 + 360s + 204s^2 + 36s^3 + 2s^4}$$

For getting the second order model, first denominator is obtained using differentiation method as described before in Step-1

$$D_2(s) = 7.0588 + 5.2941s + s^2$$

Then factor division algorithm discussed in Step-2 is applied as

$$\frac{N(s)}{D_2(s)} = \frac{(2400 + 1800s + 496s^2 + 28s^3)}{(7.0588 + 5.2941s + s^2)} = 16941.12 + 25411.68s + 15430.5448s^2 + 4623.52s^3 + 644.2340s^4 + 28s^5$$

also

$$D(s) = 240 + 360s + 204s^2 + 36s^3 + 2s^4$$

$$\alpha_0 = 70.588 \sqrt{\begin{bmatrix} 16941.12 & 25411.68 \\ 240 & 360 \end{bmatrix}}$$

Numerator of second order model is written as

$$N_2(s) = \alpha_0 + \alpha_1s = 70.588$$

Therefore, finally second order model is given by

$$G_2(s) = \frac{70.588}{7.0588 + 5.2941s + s^2}$$

**Example 2:** Consider a sixth-order two input two output system described by the transfer matrix $G_6(s)$.

$$G_6(s) = \begin{bmatrix} \frac{2(s + 5)}{(s + 1)(s + 10)} & \frac{(s + 4)}{(s + 2)(s + 5)} \\ \frac{(s + 10)}{(s + 1)(s + 20)} & \frac{(s + 6)}{(s + 2)(s + 3)} \end{bmatrix}$$

$$= \frac{1}{D(s)} \begin{bmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{bmatrix}$$

where the common denominator $D(s)$ is given by

$$D(s) = 6000 + 13100s + 10060s^2 + 3491s^3 + 571s^4 + 41s^5 + s^6$$

and

$$a_{11}(s) = 6000 + 7700s + 3610s^2 + 762s^3 + 70s^4 + 2s^5$$

$$a_{12}(s) = 2400 + 4160s + 2182s^2 + 459s^3 + 38s^4 + s^5$$

$$a_{21}(s) = 3000 + 3700s + 1650s^2 + 331s^3 + 30s^4 + s^5$$

$$a_{22}(s) = 6000 + 9100s + 3660s^2 + 601s^3 + 42s^4 + s^5$$

The proposed method is successively applied to each element of the above transfer function matrix and the reduced order models $r_i(s)$ are obtained. The general form of second order reduced transfer function matrix is taken as

$$[D_2(s)] = \frac{1}{D(s)} \begin{bmatrix} b_{11}(s) & b_{12}(s) \\ b_{21}(s) & b_{22}(s) \end{bmatrix}$$
where,
\[ \hat{D}(s) = s^2 + 6.510934s + 8.946322 \]

and
\[ b_{11}(s) = 8.9463 - 1.5407s \]
\[ b_{12}(s) = 3.5785 + 0.9941s \]
\[ b_{21}(s) = 4.47316 - 0.99403s \]
\[ b_{22}(s) = 8.9463 + 0.54676s \]

The ISE is calculated for each element \( r_{ij}(s) \) of the transfer matrix of the lower order system \([D_2(s)]\) and it is given by
\[ E = \int_0^\infty \left[ g_{ij}(t) - r_{ij}(t) \right]^2 dt \]
where, \( i, j = 1, 2 \); and \( g_{ij}(t), r_{ij}(t) \) are the unit step responses of the original system and reduced order models respectively. The comparison of 2\(^{nd}\) order with original system is shown in Fig. 1, and the comparison of step responses is shown in Fig. 2.
Conclusions

A method which combines the advantages of the differentiation method and the factor division algorithm has been presented, to derive stable reduced order models for linear time invariant systems. In this method, the reduced order denominator polynomial is determined by differentiation method and reduced order numerator is obtained by factor division. This combined method has been extended for the order reduction of linear multivariable systems. This combined algorithm is simple and efficient. The integral square error ISE in between the transient parts of original and reduced order systems is calculated and compared in the tabular form as given in Tables 1 and 2. The proposed method preserves model stability and avoids error in between the initial or final values of the time responses of original and reduced order systems.

References


Table 1 — Comparison of reduced order models for Example 1

<table>
<thead>
<tr>
<th>Method of reduction</th>
<th>Reduced models</th>
<th>ISE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lucas13</td>
<td>( G_2(s) = \frac{40 + 30s}{4 + 6s + 3s^2} )</td>
<td>0.2061</td>
</tr>
<tr>
<td>Proposed Algorithm</td>
<td>( G_r(s) = \frac{70.588}{7.0588 + 5.2941s + s^2} )</td>
<td>1.101</td>
</tr>
<tr>
<td>Routh-Hurwitz array</td>
<td>( G_r(s) = \frac{13.043478 + 9.046283s}{1.304348 + 1.701323s + s^2} )</td>
<td>1.208</td>
</tr>
<tr>
<td>Prasad et al.17</td>
<td>( G_r(s) = \frac{1.190362 + 22.532255s}{1.190362 + 3.145997s + s^2} )</td>
<td>2.743</td>
</tr>
</tbody>
</table>

Table 2 — Comparison of reduced models for Example 2

| Method of reduction | \( r_{ij}(s) \) for \( i,j=1,2 \) |
|---------------------|-------------------|-------------------|-------------------|-------------------|
| Proposed Algorithm  | 0.046070          | 8.875×10^{-8}    | 0.01280           | 0.000397          |
| Prasad et al.19     | 0.030689          | 0.000256         | 0.261963          | 0.021683          |
| Safonov and Chiang20 | 0.590617          | 0.037129         | 0.007328          | 1.066123          |
| Prasad and Pal21    | 0.136484          | 0.002446         | 0.040291          | 0.067902          |