Computer analysis of whistler mode instability in the presence of perpendicular a.c. electric field for a Lorentzian (kappa) magnetoplasma

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The dielectric tensor, modified plasma dispersion function and dispersion relations for whistler mode instability are obtained for a generalized kappa distribution function in the presence of perpendicular a.c. electric field. Integrals and modified plasma dispersion functions are reduced to series form. Numerical methods using computer techniques are used to obtain temporal growth rate for a magnetospheric plasma. The results have been compared with observed GEOS1 and GEOS2 satellite data and HF generated whistler wave in polar electrojet region. The potential application for planetary atmosphere is also indicated.

1 Introduction

Observational studies of natural space plasmas dealing with wave-particle interaction have clearly revealed the presence of a non-Maxwellian high energy tail component that can best be modelled by a generalized Lorentzian (kappa) distributions\(^1\)\(^-\)\(^4\). The resonant interactions between particles and plasma waves often occur for velocities well above the thermal speeds. Thus, the study of whistler and ion cyclotron wave for left-handed and right-handed polarized mode waves can be modelled by a kappa distribution\(^5\). Electric field measurements at magnetospheric heights have given values of a.c. field along and perpendicular to earth's magnetic field as reported earlier by Mozer et al.\(^6\) and others\(^7\)\(^\)\(^-\)\(^8\). Evidence of electric field in the magnetosphere obtained from ISEE, GEOS and Viking satellites were given in detail by Falthammar\(^9\). Direct electric field measurements on satellites, such as S3-3 and Viking, show the fluctuations in the frequency range 0.1-100 Hz or higher\(^10\). Recently, particle and field measurements from the DMSPF8 and CRRES satellites were reported in the inner magnetosphere\(^11\). Earlier, the effects of parallel d.c. electric field on whistler mode instability in the magnetosphere were studied\(^12\)\(^-\)\(^14\) for Maxwellian plasma. In recent past, generations of whistler mode instability in variety of physical situations in the presence of parallel and perpendicular a.c. field for bi-Maxwellian as well as generalized Maxwellian plasma were made\(^15\)\(^-\)\(^16\) by characteristics methods using the perturbed and unperturbed trajectories in the presence of perpendicular a.c. electric field.

In the present paper, a generalized Lorentzian (kappa) distribution function with spectral index, \(\kappa\), is considered for background plasma using the perturbed and unperturbed particle trajectories by characteristic method. The elements of dielectric tensors and the resulting dispersion relation for whistler mode wave in the presence of perpendicular a.c. electric field are given in Sec. 2. The numerical estimate of temporal growth rate for Lorentzian (kappa) as well as bi-Maxwellian (under the limit \(\kappa\rightarrow\infty\)) plasma were made by method of computer technique instead of bringing the growth rate in analytical form using simple asymptotic expansion of unmodified dispersion function. The plasma dispersion function also gets modified in the presence of perpendicular a.c. electric field. The given integrals in elements of dielectric tensors also get modified and were solved by using the expansion of power series in the limit of \(\lambda\rightarrow 0\).

2 Dielectric tensor and dispersion relation

The generalized dielectric tensor for parallel propagating whistler waves in the presence of perpendicular a.c. electric field has been obtained by substituting the limit of \(k_\perp\rightarrow 0, k = k_0 \hat{\vec{e}}_z\), into the dielectric tensor as given by Misra and Pandey\(^16\). For whistler mode instability the elements of dielectric tensor gets simplified and dispersion relation reduces to

\[
\begin{vmatrix}
-N_\parallel^2 + \epsilon_{11} & \epsilon_{12} & 0 \\
-\epsilon_{12} & -N_\parallel^2 + \epsilon_{22} & 0 \\
0 & 0 & \epsilon_{33}
\end{vmatrix} = 0 \quad \ldots \quad (1)
\]
Equation (1) is rewritten in a more convenient form as

\[ N_1^4 - 2\varepsilon_{11} N_1^2 + \varepsilon_{11}^2 = 0 \]  

\[ \varepsilon_{33} = 0 \]  

For right- and left-handed polarized whistler waves and for electrostatic waves,

\[ (2) \]

The equilibrium particle distribution \( f_s(v) \) for the background warm plasma is assumed to be a bi-Lorentzian (kappa) which is given as

\[ f_s^K(v) = \frac{1}{\pi^{1/2} \theta_{l_k}^2 \theta_{i_k}^{3/2}} \frac{\Gamma(\kappa+1)}{\Gamma(1-\frac{1}{2})} \times \left[ 1 + \frac{v_i^2}{\kappa \theta_{i_k}^2} + \frac{v_l^2}{\kappa \theta_{l_k}^2} \right]^{-(\kappa+1)} \]  

\[ \text{with associated effective thermal speeds.} \]

\[ \theta_{l_k} = [(2\kappa-3)/\kappa]^{1/2} (T_{l_k} / m_s)^{1/2} \]

\[ \text{and} \]

\[ \theta_{i_k} = [(2\kappa-3)/\kappa]^{1/2} (T_{i_k} / m_s)^{1/2} \]

where, \( \theta_{l_k} \) and \( \theta_{i_k} \), are the respective thermal speeds parallel and perpendicular to the background magnetic field, \( \kappa \) is the spectral index (the parameter \( \kappa \) generally takes on positive integral values \( \geq 2 \)), and \( \Gamma \) the gamma function.

The use of this distribution function reduces the dielectric tensor in the presence of perpendicular a.c. field as

\[ \varepsilon_{11} = 1 + \frac{\omega_p^2}{\omega^2} \frac{1}{\kappa^2} \sum_{n=-\infty}^{\infty} n^2 \times \left\{ A_{s}^K S_1 + \left( A_{s}^{K\xi} S_{2} + \frac{\omega}{k_{\parallel} \theta_{l_k}} \right) S_2 \right\} \]  

\[ \varepsilon_{11} = \varepsilon_{22} \]  

\[ \varepsilon_{12} = \sum_{n=-\infty}^{\infty} \frac{\omega_p^2}{\omega^2} \frac{1}{\kappa^2} \sum_{n=-\infty}^{\infty} i n \times \left\{ A_{s}^K S_1 + \left( A_{s}^{K\xi} S_{2} + \frac{\omega}{k_{\parallel} \theta_{l_k}} \right) S_2 \right\} \]  

\[ \text{The integrals } S_1, S_2, \ldots S_4 \text{ reduce to series form in the limit of } \lambda \to 0 \text{ and } k_{\parallel} \to 0 \text{ as given by Summers et al.} \]

\[ S_1 = S_3 = \frac{\lambda^2}{2} + \ldots \]

\[ S_2 = S_4 = \frac{\lambda^2}{2} \left( \frac{\kappa-1}{2} \right) \left( \frac{\kappa-1}{\kappa} \right)^{1/2} \times \left[ \left( \frac{\omega - \omega_{ci} + v}{k_{\parallel} \theta_{l_k}} \right) + \ldots \right] \]

\[ + \frac{\omega_{ci}}{(\omega_{ci}^2 - v^2)} \left( \frac{\lambda^2}{2} \left( \frac{\kappa-1}{2} \right) \left( \frac{\kappa-1}{\kappa} \right)^{1/2} \times \left[ \left( \frac{\omega - \omega_{ci} + v}{k_{\parallel} \theta_{l_k}} \right) + \ldots \right] \]

\[ \text{with } \xi = \frac{\omega - \omega_{ci} + v}{k_{\parallel} \theta_{l_k}}, \quad \Gamma = \frac{e_s E_0}{m_s} \]

The function \( Z_{k-1}^*(\xi) \) occurring in Eq. (9) is the modified plasma dispersion function with changed \( \xi \) and is given by Summers and Thorne as

\[ Z_{k-1}^*(\xi) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\kappa+1)}{\kappa^{1/2} \Gamma(\kappa-\frac{1}{2})} \times \left[ \frac{dt}{(t - \xi) \left( 1 + \frac{r^2}{\kappa} \right)} \right]_{-\infty}^{+\infty}, \quad \text{Im}(\xi) > 0 \]

Equation (11) is valid for positive integral values of \( \kappa \), with \( \xi = x + iy \). Following Summers and Thorne for integral values of \( \kappa > 3/2 \), \( Z_{k-1}^* \) is expressed in closed form of finite series as
Substituting the values of integrals $S_1$, $S_2$, $S_3$ and $S_4$ into Eqs. (5) and (7) the desired elements of dielectric tensors are obtained. Subsequently, these dielectric tensor elements are put into Eq. (2) to obtain the resulting dispersion relation for right-handed polarized whistler mode wave propagating parallel to the magnetic field in the presence of perpendicular a.c. field. In case higher powers of $N$ is neglected in equation (2), the dispersion relation for whistler wave reduces to

$$
\frac{c^2 k^2}{\omega^2} = 1 + \sum \frac{\omega^2}{\omega^2} \left[ \frac{\omega}{k_0 \theta_{kh}} - B_2 Z^*_{k-1}(D_k \xi) \right.
+ A_1 \left[ 1 + C_2 D_k \xi Z^*_{k-1}(D_k \xi) \right]
+ \left. \frac{\sqrt{2} \Gamma_{12}}{2(\omega^2 - \nu^2) \theta_{kh}} \left[ \frac{\omega}{k_0 \theta_{kh}} B_2 Z^*_{k-1}(D_k \xi) \right.ight.
+ \left. A_1 \left[ 1 + C_2 D_k \xi Z^*_{k-1}(D_k \xi) \right] \right] \right] \quad \ldots (13)
$$

where constants

$$
B_2 = \frac{\kappa - \frac{1}{3}}{\kappa - \frac{2}{3}} \left( \frac{\kappa - 1}{\kappa} \right)^{3/2}; \quad C_2 = \frac{\kappa - 1}{\kappa - \frac{2}{3}}; \quad D_3 = \sqrt{\frac{\kappa - 1}{\kappa}}
$$

Now when the a.c. field contribution is set to zero it reduces to the form given earlier\(^{1,7,18}\). In this case the most important differences in the dispersion relation lies in the fact that the argument of Bessel function is determined by the amplitude and frequency of the external a.c. signal. The plasma dispersion function is also modified accordingly by the frequency of the a.c. signal\(^{1,6}\).

In order to compare these results with those obtained earlier for bi-Maxwellian plasma, the limit $\kappa \to \infty$ and $\lambda \to 0$ are substituted in Eqs (11) and (13), and the resulting expression reduces to

$$
\frac{c^2 k^2}{\omega^2} = 1 + \sum \frac{\omega^2}{\omega^2} \left[ \frac{\omega}{k_0 \alpha_{kh}} - Z(\xi) + A_1^M \left[ 1 + \xi Z(\xi) \right] \right.
+ \frac{\sqrt{2} \Gamma_{12}}{2(\omega^2 - \nu^2) \alpha_{kh}} \left[ \frac{\omega}{k_0 \alpha_{kh}} Z(\xi) + A_1^M \left[ 1 + \xi Z(\xi) \right] \right] \right]
$$

where, 

$$\xi = \frac{\omega - \omega_c + \nu}{k_0 \alpha_{kh}}$$

and

$$Z(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - \xi} dt, \quad \text{Im}(\xi) > 0 \quad \ldots (15)$$

Equation (14) is exactly similar to the result obtained earlier [Eq. (21) of Ref. 16]. Further, when the magnitude of a.c. field $E_0$ and frequency $\nu$ become 0 then it reduces to the well known expression reported earlier\(^{1,18,20}\).

#### 3 Results and discussion

A numerical procedure, using computer technique without any other numerical approximations, for numerical evaluation of normalized temporal growth rate for whistler waves in the presence of perpendicular a.c. electric field has been adopted. The calculations have been carried out using Eq. (13) neglecting the contribution of higher harmonics for generalized Lorentzian (kappa) distribution function suited to magnetosperic conditions. To investigate the effect of various values of a.c. magnitude, a.c. frequency and the specific values of spectral index $K$, etc. on the growth rate in magnetospheric plasma corresponding to the geostationary satellite height of $L = 6.6$, the following plasma parameters are considered with $B_0 = 10^{-7}$ Tesla at the equatorial height. Plasma parameters $T_\parallel = 5$ keV; temperature anisotropy, $A_T^* = \left( \frac{T_\parallel}{T_\parallel - 1} \right)$ = 0.25, 0.50, 0.75, 1.0, 1.25, 1.50, 1.75, 2.00; $E_0 = 10$, 20, 30 mV/m; a.c. frequency $\nu = 2, 4, 6, 8$ kHz and $N_0 = 1 \times 10^6$ m$^{-3}$ and $K = 2, 4, \ldots, \infty$ etc. The temporal growth rate for the real values of wave vector $k$ and complex wave frequency $\omega$, is solved numerically. The function $Z^*_{k-1}(\xi)$ occurring in integrals $S_2$ and $S_4$, may be calculated numerically by the property of $Z^*_{k}(\xi)$ as power series.
under the restricted conditions $\kappa \geq n + \frac{1}{2}$ and $|\xi| \to \infty$, as is given in expansion of $S_1$, $S_2$, $S_3$ and $S_4$ for $\lambda \to 0$ and for cyclotron harmonic $n = 1$. According to this choice of plasma parameters, the explanations and details of results are given as follows.

Figure 1[(a) and (b)] shows the variation of normalized temporal growth rate $\bar{\gamma}(=\frac{\omega_i}{\omega_c})$ with the variation of normalized wave number $\bar{K} = \left(\frac{k_10_k}{\omega_c}\right)$ for the various magnitude ($E_0$) of the perpendicular a.c. signal for Lorentzian (kappa) plasma ($\kappa = 2$) and for bi-Maxwellian plasma ($\kappa = \infty$). Figure 1[(a) and (b)] also shows the variation of normalized real frequency $x = (\omega_r/\omega_c)$ with normalized wave number $\bar{K}$, for various magnitudes of perpendicular $E_0$ of the a.c. signal for Lorentzian (kappa) plasma ($\kappa = 2$) as well as for bi-Maxwellian plasma ($\kappa = \infty$). In both the

![Figure 1](image-url)

**Fig. 1**—Normalized temporal growth rate $\bar{\gamma}$ (solid lines) and the normalized real frequency $x$ (dotted lines) of the whistler instability as functions of normalized wave number $\bar{K} = \left(\frac{k_10_k}{\omega_c}\right)$ for various values of a.c. field magnitude ($E_0$) in the case of (a) Lorentzian (kappa) plasma for $\kappa = 2$, and (b) Maxwellian plasma when $\kappa = \infty$. 
cases, i.e. for Lorentzian (kappa) plasma ($\kappa = 2$) as well as bi-Maxwellian plasma ($\kappa = \infty$), the change in magnitude of perpendicular $E_0$ of a.c. signal has been found to have a nominal effect on the growth rate. The value of growth rate is more for Lorentzian plasma than that for Maxwellian plasma. The frequency does not change in both the cases. A further increase in magnitude of perpendicular a.c. field, the maxima of growth rate and the real frequency do not get affected. Thus, a minimum of perpendicular a.c. field magnitude is enough to trigger the whistler emission. A further increase of an externally applied perpendicular a.c. field is not needed. These triggered emissions have been observed by instruments on board satellites$^{23,24}$ and if these emissions are ducting along the field lines, they may be recorded by ground stations.

Figure 2[(a), (b)] shows the variation of normalized temporal growth rate $\tilde{\gamma}$ with normalized wave vector $\vec{K}$ for whistler mode instability at equatorial height.

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Fig. 2—Normalized temporal growth rate $\tilde{\gamma}$ (solid lines) and the normalized real frequency $x$ (dotted lines) of the whistler instability as functions of normalized wave number $\vec{K} = (k_1 \phi_1 / \omega_c)$ for various values of a.c. field frequency ($\nu$) in the case of (a) Lorentzian (kappa) plasma for $\kappa = 2$, and (b) Maxwellian plasma when $\kappa = \infty$. 
for various values of perpendicular a.c. field frequencies having an amplitude of $E_0 = 20$ mV/m and temperature anisotropy $A_T^* = \left( \frac{T_\perp}{T_\parallel} - 1 \right) = 1.0$ for both the cases of Lorentzian (kappa) plasma at $\kappa = 2$, and bi-Maxwellian plasma at $\kappa = \infty$. Figure 2 also shows the variation of normalized real frequency ($\chi$) with normalized wave vector $\textbf{K}$. For Lorentzian (kappa) plasma at $\kappa = 2$, this gives the maximum temporal growth rate increasing by more than an order of magnitude higher than that for bi-Maxwellian plasma at $\kappa = \infty$. The maxima of temporal growth rate shifts with change in a.c. field frequency towards the higher values of wave number $\textbf{K}$, and the emitted frequency for whistler mode instability shifts towards the lower range of frequencies. Misra and Haile\textsuperscript{15,16} pointed out that the parallel electric field is contributing to the parallel resonance velocity, giving maximum emission possibilities due to interaction occurring over extended period of time, but nevertheless, the perpendicular electric field which is modifying the perpendicular velocity and which is actively contributing to the energy exchange between wave and particle also contributes significantly to the emission of VLF signals and can explain the lower-frequency side of the whistler spectrum\textsuperscript{10,25-27}. However, the effect of a.c. frequency is to trigger instabilities, but it affects the growth rate of the maxima marginally in comparison to the above source of free energy. The Doppler shift in the frequency is not affected by the magnitude of the electric field, but only by its frequency, unlike the parallel electric field case\textsuperscript{16}. The growth rate is mainly affected by the dispersion function. The dispersion function, in the case of parallel electric field, contains a.c. field magnitude and frequency as $\xi = \omega - (k_0 E_0 / \nu)$ whereas in the case of a perpendicular electric field it contains frequency only as $\xi = \omega + \nu \pm \omega_e$. The energy exchange between the electrons, the component of the wave electric field and the imposed a.c. field perpendicular to the magnetic field, mainly contribute to cyclotron growth or damping of the waves. The real frequency in Lorentzian and Maxwellian plasma does not show much variation for small frequency changes. However, the spectral index ($\kappa$) variations are significant, which is a measure of the relative number of particles in high energy tail of the distribution\textsuperscript{28}. The curves have been drawn up to a point where approximations for the whistler mode do not break down. The real frequency increases with the increase of a.c. frequency along with the growth rate. The frequency of the a.c. field brings the maxima to different $\textbf{K}$ (or frequencies), as if the resonant charged particles are oscillating at different cyclotron frequencies and absorbing energy and thus growing. The unexplained part of the lower frequencies of the VLF and ELF spectra may be explained in terms of a perpendicular energy exchange for some suitable spectral index $\kappa$.

Figure 3[(a) and (b)] gives the variation of normalized temporal growth rate $\tilde{\gamma}$ with normalized wave vector $\textbf{K}$ for various values of temperature anisotropy $A_T^* = \left( \frac{T_\perp}{T_\parallel} - 1 \right)$ having an amplitude of $E_0 = 20$ mV/m and magnitude of a.c. frequency $\nu = 4$ kHz.
for both Lorentzian (kappa) plasma at $\kappa = 2$ and bi-Maxwellian plasma at $\kappa = \infty$. The increase of temperature anisotropy increases the growth rate and shifts the maxima to lower $K$ (higher frequencies) in both the cases. The value of maximum growth rate is slightly higher in case of Lorentzian (kappa) plasma than that for bi-Maxwellian plasma. Thus, one may conclude that presence of temperature anisotropy which is the major source of providing the free energy, seeds the whistler mode instability both in Lorentzian (kappa) and bi-Maxwellian plasma. The results are in agreement with those of Gary and Cairns\textsuperscript{29}. However, in case of Lorentzian (kappa) plasma, series expansion of $\lambda = \frac{1}{2} \left( \frac{k_\perp \theta_\perp}{\omega_c} \right) \rightarrow 0$ gives results which are varying with $\lambda^2$ bringing a change in perpendicular thermal velocity $\theta_\perp = \left[ (2\kappa - 3)/\kappa \right]^{1/2} \left( T_\perp / m \right)^{1/2}$. Therefore, any increase in $\theta_\perp$ perpendicular, shall affect marginally $T_\perp$ affecting temperature anisotropy terms. Temperature anisotropy, being the primary source of instability, gets further modified by Lorentzian (kappa) distribution function. Thus, for Lorentzian (kappa) distribution there is an increase in source of energy giving rise to the increase in growth rate. Recently\textsuperscript{30}, it was found that suprathermal electron in kappa distribution modifies the intensity and Doppler frequency of electron plasma lines.

Figure 4\textsuperscript{(a) and (b)} deals with the variation of normalized temporal growth rate $\tilde{\gamma}$ with temperature anisotropy $A_T$ for various values of a.c. frequencies having fixed amplitude of $E_0 = 20$ mV/m and normalized wave number $\tilde{K} = 0.4$ both for Lorentzian (kappa) plasma at $\kappa = 2$ and bi-Maxwellian plasma at $\kappa = \infty$. The growth rate increases with the increase in magnitude of a.c. frequency for both the cases, but the limit is reached where temperature anisotropy ceases to increase the growth rate for higher values of $A_T$ (more than 1.5) and thereafter falls sharply. Analytically, in case of bi-Maxwellian plasma, the conditions given by Cuperman and Landau\textsuperscript{20} have shown the dependence of maximum growth rate on temperature anisotropy. This effect remains basically applicable to the Lorentzian (kappa) plasma also, except that the limit of temperature anisotropy in this case is little higher, because of series solution involving $\kappa$ as well as the limiting value of $\lambda$, contributing to temperature anisotropy.

![Graph](image_url)

**Fig. 4**—Dependence of the normalized temporal growth rate $\tilde{\gamma}$ on the temperature anisotropy $A_T$ of the whistler instability for various values of the a.c. field frequency ($\nu$) in the case of (a) Lorentzian (kappa) plasma for $\kappa = 2$, and (b) Maxwellian plasma when $\kappa = \infty$.

4 Summary and conclusion

The magnitude of a.c. field ($E_0$) does not affect the maximum growth rate as well as emitted frequencies either for Lorentzian (kappa) plasma or bi-Maxwellian plasma, whereas the a.c. frequencies for a fixed amplitude increases the instability ranges. It is found that in case of Lorentzian (kappa) plasma the growth rate is higher than that for bi-Maxwellian plasma. The source of instability is temperature anisotropy in general. The temperature anisotropy $A_T$ shifts the maxima towards lower value of $\tilde{K}$ in both the cases of Lorentzian (kappa) plasma as well as bi-Maxwellian plasma and thus, the generated frequencies are different in both the cases. The Lorentzian plasma adds some additional energy by increasing the perpendicular thermal velocity because of series expansion and $\kappa$ value. The inclusion of temperature anisotropy in Lorentzian (kappa) plasma...
and bi-Maxwellian plasma can explain the observed higher frequencies spectrum of whistler waves.

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