Analytical and numerical validation of the developed structural similitude for elastic models

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The structural model must be designed, loaded and interpreted according to a set of similitude requirements that relates the mode (scaled down) to the prototype (original) structure. These similitude requirements which are based upon the theory of modeling can be derived from the dimensional analysis. In the present study, an attempt has been made to develop a scaling law for predicting the response of structures. The developed similitude relations are validated with analytical and numerical methods.

Keywords: Similitude, Scaled down model, Scaling law

The scaling concept has been utilized in many engineering applications and helps engineers and scientists to replicate the behavior of the prototype. The scaling can be either scaling up or scaling down depending upon the application. The experimental results of the scaled model can be utilized to predict the behavior of the prototype. The main objective of conducting experiments on structures at reduced scales is to reduce the cost of experimentation. Cost is reduced due to the reduction in the loading equipment and a reduction in the cost of test structure fabrication and testing. The researchers must be careful and clear on how far the model behaves similar to the prototype. The modeling accuracy depends upon the model material properties, fabrication accuracy, loading techniques, measurement methods and interpretation of results. Elastic models can be easily built to give high correlation with the prototype, if the model is fabricated and loaded perfectly.

The scaling down concept is developed based on similitude theory and has been applied to different fields like structural engineering, vibration and impact problems. Simitses1 applied similitude concept for laminated plates subjected to transverse, buckling and free vibration. RezaeePazhand et al.2,3 have carried out analytical investigations on similitude theory applied to free vibrations of laminated plates. The investigations focused on the use of scaling laws for multilayered composite rectangular plates. The authors4-6 derived the scaling laws for the vibration and buckling behavior of laminated rectangular plates. Wu7 presented a technique for predicting the vibration characteristics of an elastically supported flat plate subjected to moving loads. The similarity conditions between the prototype and the scaled model were derived from the equation of motion and dimensional analysis theory. Kumar et al.8 developed two test procedures for pseudo dynamic test of scaled concrete structures. The difference in the test results due to difference in testing procedure is also addressed. Oshiro et al.10,11 derived the scaling law for structures subjected to impact loads with the use of an alternative dimensionless parameter accounting for strain rate effects. Torkamani et al.12 developed scaling laws for free vibrations of cylindrical shells with longitudinal and circumferential stiffeners using similitude theory. Based on the similarity condition, the relation between natural frequencies of a scaled model and its prototype was obtained. The approach was validated numerically and experimentally on different examples. Pairod et al.13 derived a scaling law for the vibration response of rectangular plates along with a similarity requirement was derived and validated with the experimental results. Many researchers have reported the application of scaling law for specific applications. This paper deals with similitude requirement for all elastic models irrespective of the geometry, material and loading. The applicability of scaling laws has been validated using analytical and numerical methods.
Similitude Relation

This section explains the use of Buckingham \( \pi \) Theorem in developing the geometrical relation, loading relation and material property relation between the model and the prototype. The relation between the model and prototype is termed as similitude relation. Any structural model must be designed, loaded and interpreted according to a set of similitude requirements that relate the model to the prototype. In general, for any field problem, three independent scale factors, which represent three fundamental dimensions, namely mass, length and time, need to be selected for designing the scaled model. This selection of the scale factors and the fundamental dimensions can be derived from the principle of dimensional analysis. 

The fundamental measures of any physical phenomena are commonly referred as dimensions such as length, mass, time, temperature, electric charge etc. All governing equations of physical problems can be expressed as

\[ f(x_1, x_2, \ldots, x_k) = 0 \quad \ldots (1) \]

where \( x_1, x_2, \ldots, x_k \) are \( k \) physical variables

Eq. (1) can be expressed in the form of

\[ g(\pi_1, \pi_2, \ldots, \pi_r) = 0 \quad \ldots (2) \]

where \( \pi_1, \pi_2, \ldots, \pi_r \) are dimensionless products of the \( k \) physical variables, where \( r \) is the number of fundamental dimensions \((F, L, T)\) that are involved in the physical problem (physical variable).

According to similitude requirement, the \( \pi \) terms \((\pi_1, \pi_2, \ldots, \pi_r)\) must be equal in the model and the prototype, in order to match the functional relationship between them. Dimensional analysis provides substantial benefit in the investigation of physical behavior of any system because it permits the experimenter to combine the variables into convenient groups (Pi terms).

The procedure to find the dimensionless groups through this theorem is as follows: (i) list all the \( k \) variables involved in the problem, (ii) decompose the variables in terms of the basic dimensions \((F, L, T)\), (iii) determine the number of \( \pi \) terms, (iv) determine the repeating and non repeating variables, (v) form a \( \pi \) term for each non-repeating variable forming a relation with the repeating variables, (vi) check that each \( \pi \) term is dimensionless and (vii) write the functional relation between the \( \pi \) terms.

The criteria for selecting the repeating variables are as follows: (i) the repeating variables must be an independent variable, (ii) it should not form a dimensionless group, (iii) no two repeating variables should have same dimensions and (iv) selection, generally includes geometric parameters, field parameters and material properties.

Consider a cantilever beam shown in Fig. 1 with Young’s modulus \((E)\), length \((l)\), depth \((h)\), width \((w)\) subjected to a transverse point load \((P)\) at its free end. Assuming that one has a good insight into the nature of this problem, it would be apparent that the deflection ‘\( u \)’ is a function of \( P, E, l, w \) and \( h \).

The general equation to be split into dimensionless groups is given by

\[ f(u, P, E, l, w, h) = 0 \quad \ldots (3) \]

It has been observed that the total number of variables \((k)\) in Eq. (3) is six and the number of basic dimensions \((r)\) observed is two \((F, L)\). Hence, the number of dimensionless \( \pi \) groups possible is 4. The repeating variables have been chosen based on the selection criteria that the variables must represent the geometric and material properties and they together should not form a dimensionless group. The non repeating variables selected are \( u, P, w \) and \( h \). The repeating variables are \( E \) and \( l \) which represent the material property and geometry, respectively.

The first non-dimensional group \( \pi_1 \) is derived with any one non repeating variable forming a relation with the repeating variables as

\[ \pi_1 = u \left( \frac{E}{l^2} \right) \quad \ldots (4) \]

The variables in Eq. (4) are then represented in terms of their fundamental dimensions \((F, L, T)\) as Eq. (5)

\[ F^0 L^0 T^0 = L \left( F L^{-2} \right)^0 L^0 T^0 \quad \ldots (5) \]

Equating the coefficients on both sides of the equation, we derive the first dimensionless group from Eq. (4) as

\[ \pi_1 = \frac{u}{l} \quad \ldots (6) \]

Similarly the other three dimensionless groups are derived as Eq. (7).

![Fig. 1 — Cantilever beam for test problem](image-url)
\[ \pi_2 = \frac{P}{EI_l^2}; \pi_3 = \frac{h}{l}; \pi_4 = \frac{w}{l} \quad \ldots (7) \]

The four dimensionless terms derived must be equal for the model and the prototype in order to match the functional relationship between them. The first dimensionless term \( \pi_{l_{\text{model}}} = \pi_{l_{\text{prototype}}}, \) i.e.,

\[ \frac{u_m}{l_m} = \frac{u_p}{l_p}; \quad \frac{u_m}{l_m} = \frac{u_p}{l_p}; \quad \frac{u_m}{l_m} = \frac{u_p}{S}; \]

where \( S = l_p/l_m \) is the dimensional scale factor.

From the above, it follows that the model displacement is scale factor \((S)\) times lesser than the displacement in the prototype.

Similarly from the second dimensionless term,

\[ \left( \frac{P}{EI_l^2} \right)_{\text{model}} = \left( \frac{P}{EI_l^2} \right)_{\text{prototype}} \]

\[ \frac{P_m}{E_{m}l_m^2} = \frac{P_p}{E_p l_p^2} \]

where the suffix \( m \) and \( p \) denote model and prototype respectively.

Rearranging the terms we get,

\[ P_m = \frac{P_p}{S E S^2} \quad \ldots (8) \]

Thus the scale factor for the concentrated load is derived as \( S E S^2 \), where \( S E \) is the ratio of modulus of elasticity of the prototype to that of the model. The methodology discussed in the previous section is adopted for different test problems\(^{15} \) and the similitude relation for geometrical parameters, material properties and loadings were derived as given in Table 1.

For a scaled down model, say for 1/10\(^{\text{th}}\) model, the scale factor \( S = 10 \). Then the geometry of the model should be scaled by a factor \( S \), i.e., \( l_m = l_p/S \). where \( l_m \) and \( l_p \) are one of the geometrical dimensions of the model and prototype. Similarly, the displacement of model will be \( S \) times less than the corresponding displacement in the prototype.

When the model and the prototype are made of the same material, the stress field will be the same in both of them, while the displacement in the model will be \( S \) times less than the corresponding displacement in the prototype. If the model and prototype are of different materials, then the stress value in the model will be \( S E \) times less than the corresponding value in the prototype. The similitude relation for loads provides one major advantage on reduced scale elastic models. The concentrated loads are reduced from prototype loads by the factor \( S E S^2 \). This factor will be very large for small scale plastic model with \( S E \) approximately 75 for steel prototype; thus the above product becomes large, resulting in extremely small loads in the model.

### Validation of the Similitude Relation

#### Analytical validation

Consider a cantilever beam of rectangular cross-section with width, \( w = 100 \) mm, height, \( h = 10 \) mm and length, \( l = 1000 \) mm subjected to various types of loads as given in Table 2. The beam is made up of steel with Young’s modulus 200 GPa and density \( 7.850 \times 10^{-6} \) kg/mm\(^3\). Table 2 describes the effect on the cantilever beam after scaling down the beam and compares the corresponding value with that of the original model.

#### Numerical validation

An attempt has been made to apply the scaling laws to a complex structure rather than simple test problems. As a case study, a roof slab of a nuclear reactor has been chosen for the study. Roof slab is the

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Scale factor</th>
<th>Parameter</th>
<th>Scale factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimension ((h_p = \text{height or } t_p = \text{thickness}))</td>
<td>( S )</td>
<td>Mass of structure (kg)</td>
<td>( \rho_{\text{ahl}} )</td>
</tr>
<tr>
<td>Area ((A_p))</td>
<td>( S^2 )</td>
<td>Maximum deflection, ( \delta ) (mm)</td>
<td>( P/E )</td>
</tr>
<tr>
<td>Volume ((V_p))</td>
<td>( S^3 )</td>
<td>Maximum stress, ( \sigma ) (MPa)</td>
<td>( P/A )</td>
</tr>
<tr>
<td>Linear displacement ((U_p))</td>
<td>( S )</td>
<td>Cantilever beam with an axial load ( P = 1000 ) N at its free end</td>
<td>( 7.85 \times 10^3 )</td>
</tr>
<tr>
<td>Moment of inertia ((I_p))</td>
<td>( S^2 )</td>
<td>( 0.1 ) ( \times 10^{-3} ) ( 5 \times 10^{-4} ) ( 1 ) ( 1 )</td>
<td></td>
</tr>
<tr>
<td>Point load ((F_p))</td>
<td>( S_E S^2 )</td>
<td>Cantilever beam with a transverse point load of 10 N at its free end</td>
<td>( 7.85 \times 10^3 )</td>
</tr>
<tr>
<td>Line load ((F_L))</td>
<td>( S_E S )</td>
<td>( 0.1 ) ( \times 10^{-3} ) ( 2 ) ( 0.2 ) ( 6 ) ( 6 )</td>
<td></td>
</tr>
<tr>
<td>Uniformly distributed surface load ((P_p))</td>
<td>( S_E )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shear force ((V_p))</td>
<td>( S_E S^2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Moment ((M)) or torque ((T))</td>
<td>( S_E )</td>
<td></td>
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**Table 2 — Validation of similitude relation**
top cover for the main vessel in the nuclear reactor which forms a biological and thermal shielding in the axial direction. In addition to this, the roof slab supports various components in the reactor such as pumps, heat exchangers, control plug and main vessel. The net load acting on the roof slab includes the weight of all the above components and its self weight which is nearly 3800 tonnes. All the component loads are applied at the respective mounting locations. Figure 2 shows the solid model of the roof slab model under study and the detailed description of the roof slab has been presented by Ramu et al.¹⁶

One portion of the roof slab called ‘top plate’ shown in Fig. 3 is considered for the study. The scale factor used for scaling down the prototype dimension to the model dimension has been chosen as, \( S = 10 \). The prototype of the roof slab is made up of steel and the material for the corresponding model is perspex with Young’s modulus 205 GPa and 2.8 GPa respectively. As the elastic modulus of model material is approximately 73 times lesser than that of the prototype material, the stiffness of the model will be relatively low which in turn increases the deflection of the structure. The similitude relation has been formulated to maintain same deflection in the scaled model irrespective of the material of the model. This is achieved by reducing all loads by a factor \( S_E \), which is the ratio of Young’s modulus of material; in this case, \( S_E = 73.2 \).

The various similitude relations discussed earlier have been incorporated in the scaled roof slab model. Static analysis of the roof slab under component loading was performed in finite element analysis package ANSYS. Critical parameters like deflection and stress are extracted from the model after performing the numerical analysis. The deflection and stress in the prototype are derived from the corresponding values in the model through the similitude relation. As discussed earlier, the deflection in the prototype will be scale factor ‘10’ times higher than that of the model and the stress will be \( S_E = 73.2 \) times higher. The deflection in the scaled model has been extracted at four points namely A, B, C and D as highlighted in Fig. 3. As the loading is symmetric about two planes of the roof slab model, the deflection has been obtained in only one quarter section of the model.

The above deflection values will be useful to derive the corresponding values for the prototype. The deflection in the prototype should be ‘10’ times higher than that of the model deflection. Table 3 presents the comparison of prototype deformation obtained numerically with the prototype deformation derived from model deformation using similitude relation. The deformation results yield very good correlation for the similitude relation.

In order to ensure the applicability of scaling law for stress, the stress values at fifteen locations were compared. The fifteen locations chosen for comparison are referred by numbers 1 to 15 as shown in Fig. 3. Table 4 shows the stress value in the fifteen locations in the model and the corresponding values for the prototype have been derived by multiplying the model stress with the factor \( S_E = 73.2 \). The derived stress value of prototype using similitude relation is compared with numerical results.

From Tables 3 and 4 it can be observed that there is a good agreement between the results obtained through similitude method and that obtained from

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Fig. 2 — Roof slab model considered for the similitude study

Fig. 3 — Locations for comparison of deflection and stress
In general, the percentage of deviation in stress value is nearly 3% near the support locations and even lesser in locations away from the support. A maximum value of 6% has been noted in one of the locations, the reason being the difference in mesh density between the larger prototype and the smaller model. The difference in mesh density is due to the adoption of equal number of elements in both the prototype and the model. From the above analysis, it has been established that the similitude relation derived for stress as well as deformation for the model made with a different material is valid.

### Conclusions

Employing small scale models is of great importance in designing complex structures. Applying similitude theory to establish similarity among structural systems can save considerable time, provided the proper scaling laws are formulated and validated. In this study, the establishment of similarity conditions, based on Buckingham $\pi$ Theorem, is discussed and their use in the scaled model is also presented. Also the scale factor for the applied load and the corresponding interpretation of the results were also discussed. The developed relations were also validated by using both analytical and numerical methods.

### References