Couette flow of ferrofluid under spatially uniform sinusoidally time-varying magnetic fields

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The authors analysed the behaviour of Couette flow of a ferrofluid under spatially uniform sinusoidally time varying magnetic fields. The imposed magnetic field $H_z$ and magnetic flux density $B$, are spatially uniform and are imposed on the system by external sources. The governing linear and angular momentum conservation equations are solved for flow and spin velocity distributions for zero and non-zero spin viscosities as a function of magnetic field strength, phase, frequency, direction coordinates along and transverse to the plates, as a function of pressure gradient along the plates, vortex viscosity, dynamic viscosity and ferrofluid magnetic susceptibility. Solutions for certain limiting cases are also given.

1 Introduction

Technologically, the topic of asymmetric stress in magnetic fluids is intriguing because of its possible applications in the control of heat or mass transfer processes in convective flows, and in the modification of drag in boundary layers. Certain polymers and liquid crystals are expected to display these effects to a small degree, but the magnitude of the effect is large in ferrofluids. In addition, the subject holds inherent scientific interest.

Occurrence of asymmetric stress in magnetic fluids can be due to the influence of unsteady magnetic fields such as alternating, travelling or rotating magnetic fields. Okubo et al.\(^1\) studied the stability of the surface of a magnetic fluid layer under the influences of vertical alternating magnetic fields experimentally and analytically. Propagation of waves on the free surface of magnetic fluid under traveling magnetic field is experimentally investigated by Kikura et al.\(^2\). It is found that the surface velocity of the magnetic fluid depends on the intensity and frequency of travelling magnetic fields and depth of the channel contained with the magnetic fluid.

The motion of ferrofluid in a travelling wave magnetic field has been paradoxical as many investigators find a critical magnetic field strength below which the fluid moves opposite to the direction of the travelling wave while above, the ferrofluid moves in the same direction\(^3\). The concentration of the suspended magnetic particles, frequency and the fluid viscosity influence the value of the critical magnetic field. Under ac magnetic fields, fluid viscosity acting on the magnetic particles suspended in a ferrofluid causes the magnetization $M$ to lag behind a travelling $H$.

A body torque density

$$T = \mu_0 (M \times H)$$

arises when $M$ is not collinear with a spatially varying magnetic field $H$. Under this condition, Rosensweig\(^4\) has developed fluid mechanical analysis to extend traditional viscous fluid flows to account for the non-symmetric stress tensor that results when $M$ and $H$ are not collinear and to then simultaneously satisfy linear and angular momentum conservation equations for the ferrofluid.

Ferrofluid motion under travelling wave magnetic fields with sinusoidal time and space dependence has been recently analysed in the study\(^5\) wherein, the magnetic fields were non-uniform and force and torque densities were non-zero.

Recently Zahn and Greer\(^6\) have analysed ferrohydrodynamic pumping in a spatially uniform sinusoidally time-varying magnetic fields. But so far, no investigation has been made on the study of Couette flow of ferrofluid under spatially uniform sinusoidally time-varying magnetic fields when the
torque density is non-zero but the force along the walls is zero which forms the present work.

2 Formulation of the Problem

Consider the flow of an incompressible, viscous, ferrofluid confined between two parallel rigid plates, the upper plate moving with a constant speed and the lower plate being at rest. The schematic of the problem is shown with the help of Fig. 1.

For the planar layer of ferrofluid here, the flow velocity can have only the z-component and spin velocity can have only y-component and both vary with respect to the variable x, as:

\[ v = v_y(x) \hat{k}, \quad w = w_z(x) \hat{j} \]  

The imposed magnetic field \( H_z \) and magnetic flux density \( B_z \) are spatially uniform and are imposed on the system by external sources. Because the imposed fields are uniform with the y- and z-coordinates, field components can only vary with the x-coordinate. Gauss’s law for the magnetic flux density and Ampere’s law for the magnetic intensity with zero current density require the imposed fields to be uniform throughout the ferrofluid.

There are no y-component of the magnetic fields. The planar ferrofluid layer between two rigid walls is magnetically stressed by a uniform \( z \)-directed magnetic field \( H_z \) and uniform \( x \)-directed magnetic flux density \( B_x \), both of which vary sinusoidally in time at frequency \( \omega \). Therefore, the total magnetic field \( H \) and magnetic flux density \( B \) inside the ferrofluid layer are of the form:

\[ B = R[(\hat{H}_x, \hat{H}_y(x) \hat{k})e^{i \omega t}] \]  

\[ H = R[(\hat{H}_x(x) \hat{i} + \hat{H}_y \hat{k})e^{i \omega t}] \]  

and

\[ B = \mu_0 (H + M) \]  

3 Equations of the Problem

Since the fluid is incompressible:

\[ \nabla \cdot v = 0, \quad \nabla \cdot \omega = 0 \]  

and the coupled linear and angular momentum conservation equations for force density \( f \) and torque density \( T \) for a fluid in a gravity \(-g\hat{i}\) are:

\[ \rho \left[ \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right] = -\nabla p + f + 2\zeta \nabla \times \omega + (\xi + \eta)\omega^2 v - \rho g \hat{i} \]  

\[ I \left[ \frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega \right] = T + 2\zeta (\nabla v - 2\omega) + \eta \nabla^2 \omega \]  

where \( \rho \) is the mass density, \( p \) is the pressure, \( \xi \) is the vortex viscosity, \( I \) is the moment of inertia density, \( \eta \) is the dynamic viscosity and \( \eta' \) is the shear coefficient of spin-viscosity.

These equations are applied to the planar ferrofluid layer confined between two parallel walls, lower one being at rest and upper one moving with constant speed. It is assumed that the planar ferrofluid has viscous-dominated flow, so that inertia is negligible and is in steady state, so that the fluid responds only to the time average magnetic force and torque densities.

The magnetization relaxation equation with a ferrofluid undergoing simultaneous magnetization and reorientation due to fluid convection velocity \( v \) and particle spin at angular velocity \( \omega \) is:

\[ \frac{\partial M}{\partial t} + (v \cdot \omega)M - \omega \times M + \frac{1}{\tau}[M - \chi_0 H] = 0 \]  

where \( \tau \) is a relaxation time constant, \( M = (M_x, 0, M_z) \) is the magnetization vector, \( \chi_0 \) is the effective magnetic susceptibility which in general can be magnetic field-dependent but in this work will be taken to be constant and

\[ \dot{M}_x = R[\dot{M}_x e^{i \omega t}], \quad \dot{M}_z = R[\dot{M}_z e^{i \omega t}] \]

3.1 To obtain \( \dot{M}_x \) and \( \dot{M}_z \)
\( \dot{M}_x \) and \( \dot{M}_z \) are obtained with the help of Eqs (2), (3) and (8). These values are utilized in future in the coupled linear and angular momentum conservation equations.

Substituting Eqs (2) and (3) into Eq. (8) relates the magnetization components to the magnetic field \( \mathbf{H} \) as:

\[
\begin{align*}
\hat{\dot{M}}_x &= \frac{-\alpha_x \hat{M}_x + \frac{\dot{\hat{H}}_x}{\tau}}{\frac{\hat{H}_x}{\mu_0} z_0} \quad \cdots (9) \\
\dot{\hat{M}}_z &= \frac{-\alpha_z \hat{M}_z + \frac{\dot{\hat{H}}_z}{\tau}}{\frac{\hat{H}_z}{\mu_0} z_0} \quad \cdots (10)
\end{align*}
\]

where the second term in Eq. (8) has zero contribution because \( v \) is only \( z \)-directed and \( M \) can only vary with \( x \). By taking the \( x \)-component of Eq. (4) and solving for \( \psi \), one can obtain:

\[
\hat{\dot{H}}_x = \frac{\hat{B}_x}{\mu_0} - \dot{M}_x 
\]

Solving for \( \dot{M}_x \) and \( \dot{M}_z \) by using Eq. (11) in Eqs (9) and (10):

\[
\begin{align*}
\dot{M}_x &= \frac{x_0 \left[ \hat{H}_x \omega_x \tau + (i \Omega \tau + 1) \frac{\hat{B}_x}{\mu_0} \right]}{(\omega_x \tau)^2 + (i \Omega \tau + 1)(i \Omega \tau + 1 + x_0)} \quad \cdots (12) \\
\dot{M}_z &= \frac{x_0 \left[ (i \Omega + 1 + x_0) \hat{H}_z \omega_x \tau - \frac{\hat{B}_x}{\mu_0} \right]}{(\omega_x \tau)^2 + (i \Omega \tau + 1)(i \Omega \tau + 1 + x_0)} \quad \cdots (13)
\end{align*}
\]

Eqs (12) and (13) represent the magnetization of the ferrofluid as a function of the imposed fields \( \hat{H}_z \) and \( \hat{B}_x \) and as a function of the not yet known spin velocity \( \omega_x \) which can vary with position \( x \). The magnetization gives rise to a torque on the ferrofluid which causes fluid motion and thus a non-zero \( \omega_x \). The resulting \( \omega_x \) then changes the magnetization. There is a strong magneto-mechanical coupling, so that the magnetization and mechanical equations need to be self-consistently satisfied.

### 3.2 Magnetic force and torque densities

For \( 0 < x < d \), the magnetic force density is given by the equation:

\[
f = \mu_0 (M \cdot \omega) \mathbf{H} \quad \cdots (14)
\]

Solving for the \( x \)- and \( z \)-components with field components only constant or varying with \( x \), one can get:

\[
f_x = -\frac{d}{dx} \left( \frac{1}{2} \mu_0 M_x^2 \right) \quad f_z = 0 \quad \cdots (15)
\]

The time average components of the magnetic force density are then:

\[
< f_x > = -\frac{d}{dx} \left( \frac{1}{4} \mu_0 |M_x|^2 \right) \quad < f_z > = 0 \quad \cdots (16)
\]

Similarly, the torque density is given by the equation:

\[
T = \mu_0 (M \times \mathbf{H}) = -\mu_0 (-M_x H_z + M_z H_x) \quad \cdots (17)
\]

The torque is only \( y \)-directed:

\[
T_y = M_z B_x - \mu_0 M_x (H_x + M_y) \quad \cdots (18)
\]

The time average component of torque density is then:

\[
< T_y > = \frac{1}{2} \mu_0 (\dot{M}_x (\dot{H}_z + \dot{M}_y)) \quad \cdots (19)
\]

with superscript asterisks indicating the complex conjugate of a complex amplitude field quantity.

### 3.3 Coupled dimensionless linear and angular momentum conservation equations

It is convenient to express modified pressure as:

\[
p' = p + \frac{1}{4} \mu_0 M_x^2 + \rho g x \quad \cdots (20)
\]

So that Eqs (6) and (7) in the negligible inertia, viscous dominated limit become:

\[
(\zeta + \eta) \frac{d^2 v_z}{dx^2} + 2 \zeta \frac{dv_y}{dx} - \frac{\partial p'}{\partial z} = 0 \quad \cdots (21)
\]

\[
\eta' \frac{d^2 \omega_x}{dx^2} - 2 \zeta' \left( \frac{dv_y}{dx} + 2 \omega_y \right) + < T_y > = 0 \quad \cdots (22)
\]

It is convenient to express parameters in dimensionless form indicated with tildes, with time normalized to the magnetic relaxation time \( \tau \), space normalized to the distance between the walls \( d \) and magnetic field quantities normalized to a magnetic field strength \( H_0 \). Therefore:
\[ \vec{Q} = \Omega \tau, \vec{B} = \vec{M} = \frac{\vec{M}}{H_0}, \vec{T} = \frac{T_0}{\mu_0 H_0}, \vec{\omega} = \omega \tau \]

\[ \vec{T}_y = \frac{T_y}{\mu_0 H_0}, \vec{\omega} = \frac{2\eta}{\mu_0 H_0 \tau}, \vec{\omega}_y = \frac{\eta_y}{\mu_0 H_0 \tau} \]

\[ \vec{T} = \frac{2\vec{z}}{\mu_0 H_0 \tau} \frac{\partial^2 \vec{\omega}}{\partial \tau}, \vec{U} = \frac{U_0}{U_y} \] (23)

Hence the dimensionless flow and spin velocity equations are:

\[ \frac{1}{2} \left( \vec{z} + \vec{\eta} \right) \frac{d^2 \vec{v}_z}{d \tau^2} + \vec{z} \frac{d \vec{\omega}_y}{d \tau} - \frac{d \vec{\omega}}{d \tau} = 0 \] (24)

\[ \eta \frac{d^2 \vec{\omega}}{d \tau^2} - \vec{z} \left( \frac{d \vec{\omega}}{d \tau} + 2 \vec{\omega}_y \right) + < \vec{T}_y > = 0 \] (25)

where

\[ < \vec{T}_y > = \frac{1}{R} [\vec{M} \cdot \vec{B}_y - \vec{M}_y (\vec{H}_y + \vec{M}_y)] \] (26)

\[ \vec{M}_x = \frac{\chi_0 [\vec{M} \cdot \vec{B}_y + (\vec{M}_y + 1) \vec{B}_y]}{\partial_y^2 + (\vec{M}_y + 1)(\vec{M}_y + 1 + 1)} \] (27)

\[ \vec{M}_z = \frac{\chi_0 [\vec{M} \cdot \vec{B}_y - \vec{B} \cdot \vec{M}_y]}{\partial_z^2 + (\vec{M}_y + 1)(\vec{M}_y + 1 + 1)} \] (28)

This set of equations describes the motion of the planar ferrofluid layer confined rigid plates with imposed spatially uniform, sinusoidally time-varying x- and z-directed magnetic fields. The primary complexity of the analysis is that the time average torque density \(< \vec{T}_y >\) depends in a complicated way on the spin velocity \(\vec{\omega}_y\) which depends on \(\vec{x}\).

Substituting Eqs (27) and (28) into Eq. (26) gives:

\[ < \vec{T}_y > = \frac{\chi_0}{2} \left[ \vec{M}_y \cdot \vec{B}_y \right] (\vec{\omega}_y^2 - \vec{\omega}_y^2 + 1) + 1 \vec{H}_y \vec{\omega}_y \left( \vec{\omega}_y^2 - \vec{\omega}_y^2 + (1 + \chi_0) \right) \]

\[ + 2R \left[ \left( \chi_0 \right) (\vec{\omega}_y^2 - \vec{\omega}_y^2 + 1) + \vec{\omega}_y (\vec{\omega}_y^2 - \vec{\omega}_y^2 + 1 - \chi_0) \right] \right] \]

Note that the phase relationship between \(\vec{H}_y\) and \(\vec{B}_y\) is very important because, for our case studies with \(\vec{\eta} = 0, \vec{\omega}_y\) must be zero at the fixed boundaries at \(x=0, d\). It is useful to examine Eq. (29) in the limit of small \(\vec{\omega}_y\). To first order \(\vec{\omega}_y\) reduces to:

\[ \lim_{\vec{\omega}_y \to 0} < \vec{T}_y > = \vec{T}_0 + c \vec{\omega}_y \] (30)

where

\[ \vec{T}_0 = -\chi_0 R \left[ \left( \chi_0 \vec{\omega}_y^2 + i \vec{\omega}_y \vec{\omega}_y^2 + 1 + \chi_0 \right) \vec{H}_y \vec{B}_y \right] \left( \vec{\omega}_y^2 + 1 + \chi_0 \right) \] (31)

\[ \alpha = \frac{\chi_0 \left[ \vec{B} \vec{\omega}_y^2 + (\vec{\omega}_y^2 - 1) + \vec{H}_y \vec{\omega}_y \left( \vec{\omega}_y^2 - (1 + \chi_0) \right) \right]}{2 \left( \vec{\omega}_y^2 + 1 + \chi_0 \right) \vec{\omega}_y^2} \] (32)

Note that \(\alpha\) can be positive or negative depending on the value of \(\vec{\omega}_y\) compared with 1 or 1 + \(\chi_0\). It is inferred that \(\alpha\) is positive at high frequencies and negative at low frequencies.

4 Solution of the Problem

4.1 Zero spin-viscosity (\(\vec{\omega}_y = 0\)) solutions

First the simple limiting case of zero spin-viscosity is examined. Solving Eqs (24) and (25), one can obtain:

\[ \vec{v}_y (\vec{x}) = \frac{1}{\eta} \frac{d \vec{\omega}_y}{d \tau} \vec{x} - \frac{1}{\eta_0} \int \vec{T}_y d \tau + k_1 \vec{x} + k_2 \] (33)

where \(k_1\) and \(k_2\) are constants of integration to be found from the boundary conditions at the rigid boundaries:

\[ \vec{v}_y (\vec{x} = 0) = 0, \vec{v}_y (\vec{x} = d) = \vec{U}_0 \] (34)

Applying these boundary conditions to the above equation, one can get:

\[ k_1 = \vec{U}_0 \frac{1}{\eta} \frac{d \vec{\omega}_y}{d \tau} + \frac{1}{\eta_0} \int \vec{T}_y d \tau, k_2 = 0 \]

So that the velocity becomes:
\[ \bar{v}_z(x) = \bar{U}_0 x + \frac{1}{\eta} \frac{\partial \bar{p}'}{\partial \bar{\xi}} \bar{x} - \bar{x} - 1 \]
\[ + \frac{1}{\eta} \left[ \bar{x} \int_0^\bar{x} \left( \bar{T}_y > \bar{x} \right) d\bar{x} - \bar{x} \int_0^\bar{x} < \bar{T}_y > d\bar{x} \right] \] \hspace{0.5cm} (35)

And it is further implied from Eq. (25) that:
\[ \bar{\omega}_y = -\frac{\bar{U}_0}{2} - \frac{1}{2\eta} \left[ \frac{\partial \bar{p}'}{\partial z} (2\bar{x} - 1) \right] \]
\[ - \frac{\bar{\xi}}{\bar{\zeta}} + \frac{\bar{\eta}}{\bar{\xi}} < \bar{T}_y > + \left[ \bar{x} \int_0^\bar{x} < \bar{T}_y > d\bar{x} \right] \] \hspace{0.5cm} (36)

It is noted that Eqs (35) and (36) are not known analytical solutions for \( \bar{v}_z \) and \( \bar{\omega}_y \) because \( < \bar{T}_y > \) varies with \( \bar{x} \) and \( \bar{\omega}_y \) varies with \( \bar{x} \). When \( \frac{\partial \bar{p}'}{\partial z} \) is large compared with the torque density \( < \bar{T}_y > \), then the flow velocity \( \bar{v}_z \) will be essentially parabolic, while the spin velocity \( \bar{\omega}_y \) will be linear with respect to \( \bar{x} \).

If the time average torque density is constant with position, then the torque has no contribution to the flow velocity as the two torque terms in Eq. (35) cancel, but there is a constant torque contribution to the spin velocity in Eq. (36). If there is no pressure gradient, \( \frac{\partial \bar{p}'}{\partial z} = 0 \), then the solution for \( \bar{\omega}_y \) in Eq. (36) is a constant, independent of \( \bar{x} \) as <\( \bar{T}_y > \) is the only constant given by Eq. (29) and \( \bar{U}_0 \). Then the solution to Eq. (35) is \( \bar{v}_z(x) = \bar{U}_0 \bar{x} \) and
\[ \bar{\omega}_y(x) = \frac{< \bar{T}_y >}{2\bar{\zeta}} - \frac{\bar{U}_0}{2} \]

The torque is constant with position if \( \bar{\omega}_y(x) \) is constant with position as given by Eq. (29).

**4.2 Non-zero spin-viscosity solutions**

(a) General solution method

A real physical system will have \( \eta' \neq 0 \), so that Eq. (24) and (25) are a fourth-order system that can be integrated numerically by the Runge-Kutta method. The boundary conditions are:

\[ \bar{v}_z(x) = 0, \quad \bar{\omega}_y(x) = 0 \] \hspace{0.5cm} (37)
\[ \bar{v}_z(x = 1) = \bar{U}_0, \quad \bar{\omega}_y(x = 1) = 0 \] \hspace{0.5cm} (38)

The Runge-Kutta method cannot directly solve two-point boundary value problem with boundary conditions at \( \bar{x} = 0 \) and \( \bar{x} = 1 \). Rather it requires specification of functions and their derivatives at \( \bar{x} = 0 \) and the system is then numerically integrated to \( \bar{x} = 1 \). Thus our procedure was to specify \( \bar{v}_z(x = 0) = 0 \) and \( \bar{\omega}_y(x = 0) = 0 \) and to guess the derivative values:

\[ D_1 = \frac{\partial \bar{v}_z}{\partial x}, \quad D_2 = \frac{\partial \bar{\omega}_y}{\partial x} \]

At \( \bar{x} = 0 \) and then integrate numerically to \( \bar{x} = 1 \) and define:
\[ F_1 = \bar{v}_z(\bar{x} = 1), \quad F_2 = \bar{\omega}_y(\bar{x} = 1) \]

Then, use Newton’s method to find best values of \( D_1 \) and \( D_2 \) to derive \( F_1 = \bar{U}_0 \) and \( F_2 \) to zero. The resulting \( \bar{v}_z \) and \( \bar{\omega}_y \) would be the solutions describing the ferrofluid motion. As a check to the numerical method, several cases are examined with \( < \bar{T}_y > \) constant and found in excellent agreement between analytical closed-form solutions of Eqs (25) and (26) to the numerical results.

(b) Zero magnetic torque density \((< \bar{T}_y > = 0)\)

If \( < \bar{T}_y > = 0 \) in Eq. (25) the flow is driven by the pressure gradient and Eqs (24) and (25) have analytical solutions of the form:

\[ \bar{v}_z(x) = \frac{\bar{x}^2}{\eta} \frac{\partial \bar{p}'}{\partial \bar{\xi}} - 2A\bar{x} - \beta(B\bar{e}^{\bar{\alpha}} - C\bar{e}^{-\bar{\alpha}}) + D \] \hspace{0.5cm} (39)
\[ \bar{\omega}_y(x) = \frac{\bar{x}}{\eta} \frac{\partial \bar{p}'}{\partial \bar{\xi}} + A + B\bar{e}^{\bar{\alpha}} + C\bar{e}^{-\bar{\alpha}} \] \hspace{0.5cm} (40)

where
\[ a = \sqrt{\frac{2\bar{\zeta} \eta}{\eta' (\zeta + \eta)}}, \quad \beta = \sqrt{\frac{2\bar{\zeta} \eta^2}{\eta(\xi + \eta)}} \]

The coefficients \( A, B, C \) and \( D \) are found by applying the zero slip boundary conditions at the plates given by Eqs (37) and (38) and they yield:
\[ A = \left[ -2A_1 \beta + A_1 \beta (e^{-\alpha} + e^{\alpha}) + U_1(e^{\alpha} - e^{-\alpha}) \right] \frac{A_2}{A} \]

\[ B = \left[ A_1(\beta - \beta e^{-\alpha} - 2) + U_1(e^{-\alpha} - 1) \right] \frac{A_2}{A} \]

\[ C = \left[ A_1(\beta - \beta e^{-\alpha} + 2) + U_1(1 - e^{-\alpha}) \right] \frac{A_2}{A} \]

\[ D = \left[ \beta U_1(e^{\alpha} + e^{-\alpha} - 2) + A_1(\beta^2 e^{\alpha} - \beta^2 e^{-\alpha} - 4\beta) \right] \frac{A_2}{A} \]

where

\[ A_1 = \frac{1}{\tilde{\eta}} \frac{\partial^2 \tilde{\eta}}{\partial x^2}, U_1 = \tilde{U}_0 - \frac{1}{\tilde{\eta}} \frac{\partial \tilde{\eta}}{\partial x} \]

\[ A_2 = 2\beta(e^{\alpha} + e^{-\alpha} - 2) + 2(e^{\alpha} - e^{-\alpha}) \]

(c) Zero pressure gradient \( \frac{\partial \tilde{\eta}}{\partial x} = 0 \)

At the opposite limit, where the flow is driven by the magnetic torque, one can take the pressure gradient to be negligibly small. Then Eq. (24) can be integrated once to yield:

\[ \frac{1}{2}(\tilde{\eta} + \tilde{\eta}_0) \frac{\partial \tilde{\eta}}{\partial x} + \tilde{\eta} \tilde{\omega}_s = C \quad \ldots(41) \]

where \( C \) is a constant. The flow solutions are then:

\[ \tilde{v}_x(\tilde{x}) = \frac{\tilde{x} D}{\gamma(\tilde{\eta} + \tilde{\eta}_0) \sinh \gamma} \left[ 2\tilde{x}(e^{\gamma} - 1) \right. \]

\[ + (e^{\gamma} - 1)(1 - e^{-\gamma}) - (e^{-\gamma} - 1) \]

\[ \left. + (e^{-\gamma} - 1) + 2\tilde{U}_0(\tilde{\eta}_0 \sinh \gamma - \sinh \tilde{\eta}) \right] \quad \ldots(42) \]

\[ \tilde{\omega}_s(\tilde{x}) = \frac{D}{\gamma \sinh \gamma} \left[ \sinh \gamma + (e^{-\gamma} - 1) \right. \]

\[ + (1 - e^{\gamma})e^{-\gamma} + \tilde{U}_0(e^{\gamma} - e^{-\gamma}) \]

\[ \left. - \gamma \tilde{\eta} \right] \quad \ldots(43) \]

where

\[ D = \frac{\tilde{x} D}{2(\gamma^2 S + 2\gamma \tilde{\eta} - a(\tilde{\eta} + \tilde{\eta}_0))} \]

where

\[ S = 1 - e^{\gamma} + e^{-\gamma} - 2 \]

\[ \gamma = \sqrt{\frac{2\gamma \tilde{\eta} - a}{\tilde{\eta}}} \]

5 Results and Discussion

In general, the results are identical to that of Zahn and Greer\(^b\) when the upper plate is at rest for all the cases.

Non-dimensional time average magnetic torque density is given by Eq. (29). In order to have the physical insight about the variation of this, time average magnetic torque density is plotted as a function of the dimensionless spin velocity for different orientations of the applied magnetic field by fixing certain values for the parameters involved.

Fig. 2 - Effect of \( \tilde{\Omega} \) on time average torque density \( <\tilde{T}_{1y}> \)

for an axial magnetic field \((H = H_0, \tilde{B}_z = 0)\)

Figs 2-4 illustrate the dimensionless time average torque density as a function of dimensionless spin velocity for various dimensionless angular frequencies for three different cases, namely (a) axial magnetic field, (b) transverse magnetic field, and (c) rotating magnetic field. It is seen that for a stationary spin velocity \( \tilde{\omega} = 0 \) the slope around \( \tilde{\omega} = 0 \) is negative at low frequencies and positive at high frequencies and that only the rotating magnetic field case has non-zero time average torque at \( \tilde{\omega} = 0 \).

Figs 5 and 6 portray the effect of speed of the upper plate overflow and spin velocities respectively. Increasing \( \tilde{U}_0 \) increases the flow...
velocity, whereas it decreases the spin velocity for the case of non-zero spin velocity.

![Flow velocity profiles for different $U_0$ when $\tilde{f} \neq 0$](image1)

![Spin velocity profiles for different $U_0$ when $\tilde{f} \neq 0$](image2)

When $\tilde{f} \neq 0$, the effect of spin viscosity, over flow velocity and spin velocity is shown with the help of Figs 7-11. It is interesting to note that the profiles for flow velocity and spin velocity are similar in both the cases of tangential magnetic field and normal magnetic field and hence the figures are
displayed here only for the case of tangential magnetic field and this trend is maintained even for the higher values of the speed of the upper plate. In addition, it is seen that the effect of moving of the upper plate is even to change the trend of profiles of both the flow and spin velocities, which are vividly seen through Figs 7 and 8. Further, it is noted that as spin viscosity increases, flow velocity profiles have
both increasing and decreasing trend, whereas spin velocity increases for increasing $\eta'$. 

![Diagram](image1)

**Fig. 11** — Effect of $\eta'$ over spin velocity profiles for a rotating magnetic field ($\vec{H} = i \mu_0 H_0$) when $\vec{U}_0 = 10.0$

![Diagram](image2)

**Fig. 12** — Flow velocity profiles for different $a$ when $<\tilde{T}_r> = 0$

Figs 9-11 illustrate the effect of spin viscosity over flow velocity and spin velocity respectively, for a rotating magnetic field. The effect of speed of the upper plate is to suppress the values in general over the flow velocity. As spin viscosity increases, flow velocity increases as well as decreases.

![Diagram](image3)

**Fig. 13** — Spin velocity profiles for different $a$ when $<\tilde{T}_r> = 0$

However, this is just the reverse for the case of tangential or normal magnetic field. Increasing $\eta'$ decreases the spin velocity and this is shown through Fig. 10.

Distinguishing the effect of the speed of the upper plate over spin velocity is noted through Fig. 11. As $\eta'$ increases, spin velocity increases, which is just the opposite of when the speed of the
upper plate is small. However, this trend is similar for the case of tangential or normal magnetic field, whatever be the speed of the upper plate (small or relatively large).

15. Increasing speed of the upper enhances the flow velocity, whereas it has the opposite trend over the spin velocity so as to decrease it and these are depicted in Figs 12 and 13. Dominating feature of the effect of the speed of upper plate is obviously displayed through Figs 14 and 15.

Fig. 15 — Spin velocity profiles for different $a$ when $U_0 = 10$ and $<T_\gamma> = 0$

Fig. 16 — Flow velocity profiles for different $\gamma$

Dimensionless flow and spin velocities for various values of the non-dimensional parameter $a$ and $U_0$ (speed of the upper plate) with zero magnetic torque density are shown through Figs 12-

Fig. 17 — Spin velocity profiles for different $\gamma$

Figs 16 and 17 disclose the significant effect due to the speed of the upper wall over the flow and spin velocities with zero pressure gradient. The effect of moving of the upper plate is to even change the nature of velocity profiles, so as to even reverse the trend for imaginary $\gamma$.

6 Conclusion

The following conclusions can be drawn:

1. In general, in the uniform magnetic field, the magnetization characteristic depends on fluid spin velocity but does not depend on flow velocity.

2. The magnetic force density along the plates is zero, while the magnetic torque density is non-zero.

3. The effect of the upper plate accelerates the flow velocity, while it decelerates the spin velocity when the spin viscosity and pressure gradient are non-zero and magnetic torque density is zero.

4. When the speed of the upper plate is relatively higher, its effect is so significant over the flow and spin velocities so as to reverse their nature for the case of zero pressure gradient.
5. When the spin viscosity is not uniform, the moving plate effect over the flow velocity is so dominant, as to accelerate it and also to alter its trend whereas its influence over spin velocity is to decelerate it.

6. It is interesting to note that this moving plate effect is so influencing and enhancing for the case of rotating magnetic field in all cases.

7. It is further seen that, under a relatively higher speed of the upper plate, the rotating field behaviour is analogous to that of transverse or axial magnetic field over spin velocity.

References