Static properties of hadrons from MIT bag model with interacting quarks

A A Rajabi
Physics Department, Shahrood University, 7-Tir Sq, PO Box 36155-316, Shahrood, Iran
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An improved MIT bag model is used to calculate the static properties of hadrons containing u, d, s and c quarks. The two key ingredients of the improved model are the effective quark-gluon coupling constant originating from color charge and the quark confinement. In this paper, the authors choose to allow residual interactions to operate so as to go beyond of asymptotic freedom and confinement. In the model used, the valence quarks interact with themselves. The authors calculate the relativistic wave-function for quarks in a scalar-vector QCD potential, analytically. Finally, vanishing of the normal component of vector current at the surface of the baryon as a boundary condition equivalent to confinement results in the static properties and the strength of Coulomb like potential parameter. This depends on the mass parameters, contrary to almost all previous versions. The calculated static properties for baryon is better than in the uncorrected versions of the model. Magnetic moments and charge radii of some hadrons are comparable well with the experiment. The agreement for the magnetic moments is surprisingly good for n, A, Σ, Σ⁺...

1 Introduction

In the MIT bag model, hadrons are considered as extended static objects localized in space in which each quark moves freely in a spherically symmetric well of infinite depth (bag). The internal structure of particles is associated with quark and gluon field variables. Because of its simplicity, the MIT bag model is rather convenient for calculating various hadronic properties. The MIT bag model possesses many desirable features inspired by QCD and relativity. However, so far, there is no derivation of the bag model from first principles. Thus, it is important to test the model in a situation other than those in which the dynamics of the model were originally formulated. The success of the first attempts to calculate the static properties of hadrons lend credence to the bag model approach to hadronic phenomenology. By using the normalized spin ½ positive parity solution of MIT bag model for studying nuclear matter saturation mechanism, based on the quark structure of the nucleon⁷ and also recently, a quark-meson coupling QMC mechanism for the saturation of nuclear was initially proposed by Guichon⁴ and generalized by Fleck et al.⁴, Saito & Thomas⁵ and Song & Su⁶. In the QMC model, the baryon is described by the static spherical MIT bag model, using the normalized ground-state for quark in the baryons.

Although the MIT bag model gives results, which are within an acceptable range that shows the correctness of its essential ingredients, obviously, has shortcomings. One of the short-coming is the neglecting of inter-quark interactions.

The MIT bag model uses free quark wave-function with no-current boundary condition in the bag's wall. Certainly, quarks interact within the bag, which will change the standard results. In the past, certain modification of the model was introduced, giving rise to better results compared to experimental values⁶⁻⁸.

In this paper, a modification is proposed by extending the model to include certain residual interaction among quarks. The relativistic wave-function is also taken.

In the model, each quark moves inside the bag in the effective field of other quarks and gluons. As in other models, the relativistic Dirac equation is considered but, with above-mentioned, effective central potential. The ground-state wave-function is calculated analytically.

Hence, the standard model is improved by using a modified ground-state for the quarks. In Table 1, the static properties of the nucleon derived in the modified model are compared with the standard bag model results and also with experiment⁶⁻⁸.
As can be seen from the Tables, the results show considerable correction over the standard results. The agreement is far better, especially, for the magnetic moment and \( g_A/g_V \), shall be used as input and derive other values given in the Tables. The improvement is rather remarkable. For example, it gives much better values for proton compared to the standard model. Further, improved results can be found in Table 2.

### Table 1 — Comparing static properties of proton in MIT bag model and model by the authors with experiment

<table>
<thead>
<tr>
<th>Property</th>
<th>MIT Bag Model</th>
<th>Model by Authors</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_q )</td>
<td>0</td>
<td>293.139±0.823 MeV</td>
<td>-100-350 MeV*</td>
</tr>
<tr>
<td>( g_A/g_V )</td>
<td>1.09</td>
<td>1.255±0.006</td>
<td>1.255±0.006</td>
</tr>
<tr>
<td>( &lt;r_m^2&gt;^{1/2} )</td>
<td>0.73 fm</td>
<td>0.802±0.003 fm</td>
<td>0.88±0.03 fm</td>
</tr>
<tr>
<td>( \mu_p )</td>
<td>1.9 n.m</td>
<td>2.643±0.012 n.m</td>
<td>2.792 n.m</td>
</tr>
<tr>
<td>( R_p )</td>
<td>1.5 fm</td>
<td>1.234±0.000 fm</td>
<td>--</td>
</tr>
<tr>
<td>( \alpha_c )</td>
<td>0</td>
<td>0.540±0.002</td>
<td>0≤( \alpha_c )≤1*</td>
</tr>
</tbody>
</table>

*The values are not directly measured, but inferred from experiment.

Now, the different interactions that are considered are briefly discussed. The effective interaction of quarks due to gluon exchange is assumed to be given by central potential for each quark. It is assumed that, in effect, each quark feels a force due two other ones, as if they were combined as one anti-quark, giving rise to the potential:

\[
V(r) = \frac{\lambda \alpha_c}{r} + f(r) \quad \text{...(1)}
\]

The corresponding short range potential which has an attractive Columbic form for small quark-anti-quark separations, originates from the color charge:

\[
V(r) \rightarrow \frac{\lambda \alpha_c}{r} \quad \text{for } r \rightarrow 0 \quad \text{...(2)}
\]

Many forms have been proposed for \( f(r) \). Assuming a quadratic term giving rise to the confining potential:

\[
V(r) \rightarrow f(r) = ar^2 \quad \text{...(3)}
\]

A good theory for the potential at intermediate separations (2) and (3) does not exist. However, a one-gluon exchange also contributes to the time component of a vector potential. Therefore, our quarks wave-function satisfy a Dirac equation with given potential.

### Table 2 — Comparing of magnetic moment of different baryons based on the MIT bag model and the model developed in this paper with experiment

<table>
<thead>
<tr>
<th>Baryon</th>
<th>MIT Bag Model</th>
<th>Model by Authors</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>-1.273 n.m</td>
<td>-1.874±0.009 n.m</td>
<td>-1.912 n.m</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>-0.494 n.m</td>
<td>-0.605±0.003 n.m</td>
<td>-0.614 n.m</td>
</tr>
<tr>
<td>( \Sigma )</td>
<td>0.684 n.m</td>
<td>-1.138±0.007 n.m</td>
<td>-1.156 n.m</td>
</tr>
<tr>
<td>( \Sigma^* )</td>
<td>+1.843 n.m</td>
<td>-2.385±0.114 n.m</td>
<td>+2.418 n.m</td>
</tr>
<tr>
<td>( \Xi^0 )</td>
<td>-1.064 n.m</td>
<td>-1.237±0.005 n.m</td>
<td>-1.256 n.m</td>
</tr>
<tr>
<td>( \Xi^- )</td>
<td>-0.437 n.m</td>
<td>-0.680±0.003 n.m</td>
<td>-0.689 n.m</td>
</tr>
</tbody>
</table>

In the following Section, by solving this Dirac equation, the relativistic wave function for valence quarks have been calculated analytically. The nucleon masses is taken and the ratio \( g_A/g_V = 1.255±0.006 \) as the input. Many authors have taken \( g_A/g_V = 1.26 \) as their input since, it is well established, experimentally. These inputs fix the parameters in the potentials and the quark mass and from them several other nucleon properties are derived. The numerical values show remarkable improvement over previous results obtained from MIT-bag model with three quarks potential, suitable for constituent quark’s masses, which are in the range of (−100 – 350) MeV.

In Section 3, the static properties of a nucleon and the strong coupling constant \( \alpha \) have been found. Finally, in Section 4 we give our conclusion.

### 2 Relativistic Wave Function

In considerations by the authors, each quark \( q \) is taken to be in a central potential \( U(r) \), which is produced by the other two quarks \( (qq) \). Hence, the single-particle wave-function satisfies the Dirac equation.

\[
\left[ \gamma_\alpha \partial + i \vec{\gamma} \cdot \vec{V} - (m + U(r)) \right] \psi(r) = 0 \quad \text{...(4)}
\]

where

\[
U(r) = \frac{1}{2} (1 + a \gamma_0) A(r) \quad \text{...(5)}
\]
The value of the parameter $a$ depends on the model\textsuperscript{1}\textsuperscript{16}. In the model of the authors, it is equal to 1. Physically, this potential would arise from an equal admixture of a scalar potential $U_0$ and the time component of a vector potential $V_0$. Using Eqs (2, 3 and 5), the potential can be written in the following form:

$$U_0 = V_0 = \frac{1}{2} A(r) = \frac{1}{2} \left( b \alpha_r + a r^2 \right) \quad \cdots (6)$$

where

$$b \alpha_r = -c \quad \cdots (7)$$

The Dirac equation form in common use for scalar potential ($U_0$) and vector potential ($V_0$) is often taken as follows:

$$\left[ i \gamma^\mu \partial_\mu - \gamma^0 V_0 - (m + U_0) \right] \psi(r) = 0 \quad \cdots (8)$$

The stationary solution reduces to:

$$\left[ i \gamma^r (\epsilon - V_0) + i \gamma^0 \nabla - (m + U_0) \right] \psi(r) = 0 \quad \cdots (9)$$

The scalar potential $U(r)$ combines with the mass and the vector potential $V(r)$ with the energy ($\epsilon$) in the Dirac equation.

The eigenvalue of Eq. (8) is denoted by $\psi_0(r)$ and rewritten:

$$\psi_{0i}(r) = \begin{bmatrix} \phi \cr \lambda \end{bmatrix} = N \begin{bmatrix} g_k(r) Y_{j}^{l_i}(r) \\ f_k(r) Y_{j}^{l_i}(r) \end{bmatrix} \quad \cdots (10)$$

Then, from Eqs (9) and (10) we get:

$$\begin{align*}
\frac{d g_k(r)}{dr} \left[ (k+1) \frac{r}{r} g_k(r) - m f_k(r) \right] & = \epsilon f_k(r) \\
\frac{d f_k(r)}{dr} \left[ 1 - k \frac{r}{r} f_k(r) - (m + A(r)) g_k(r) \right] & = \epsilon g_k(r)
\end{align*} \quad \cdots (11)$$

Now, these two equations are combined and introduced:

$$g_k(r) = \frac{1}{r} \phi_k(r) \quad \cdots (12)$$

Then we get:

$$\phi_k(r) + \left[ -k(k+1) \frac{r^2}{r^2} + \epsilon^2 - m^2 - (\epsilon + m) A(r) \right] \phi_k(r) = 0 \quad \cdots (13)$$

Assuming:

$$\lambda = \epsilon^2 - m^2, \quad a_1 = (\epsilon + m) a, \quad c_1 = (\epsilon + m) c \quad \cdots (14)$$

Then, Eq. (13) reduces to:

$$\phi_k(r) = \left[ -\lambda + a r^2 - \frac{c}{r} + \frac{k(k+1)}{r^2} \right] \phi_k(r) \quad \cdots (15)$$

An ansatz is made for the eigenfunction $\phi_k(r)$\textsuperscript{17,20}:

$$\phi_k(r) = h(r) \exp \left[ Y(r) \right] \quad \cdots (16)$$

with $h(r)$ and $Y(r)$ given by:

$$\begin{align*}
h(r) & = 1 + \alpha r, \quad \alpha \neq 0 \\
Y(r) & = \frac{1}{2} \alpha r^2 + \beta r + \delta \ln r
\end{align*} \quad \cdots (17)$$

This implies:

$$\phi_k(r) = \left[ \epsilon^2 + Y^2 + \left( \frac{h'}{h} \right)^2 \right] \phi_k(r) \quad \cdots (18)$$

The relation between energy $\epsilon$, quark mass $m$ with parameter $a_1$ and the constraint between the potential parameters $(a_1, c_1)$ turns out to be:

$$\epsilon^2 - m^2 = 5 \sqrt{a_1} \quad \cdots (19)$$

$$\sqrt{a_1} - \frac{c_1}{4} = 0 \quad \cdots (20)$$

Trying to solve this problem in the presence of center-of-mass correction, (see Appendix A)

$$(M' = M + E_{cm} = 3 \epsilon)$$

where $M'$ is the corrected nucleon mass ($M$) with centre-of-mass energy $E_{cm}$.

Let

$$x = \frac{m}{\epsilon} = \frac{3m}{M'} \quad \cdots (21)$$

the ratio of quark mass to its energy.

The potential parameters $a$ and $c$ using Eqs (14,19-21) can be found as follows:

$$a = \frac{(\epsilon + m)(\epsilon - m)^2}{25} = \frac{M'^3}{225} (1-x)^2 (1+x) \quad \cdots (22)$$
\[ c = \left[ \frac{4}{5} \frac{e-m}{e+m} \right]^2 = \left[ \frac{4}{5} \frac{1-x}{1+x} \right]^2 \] ... (23)

Eqs (10-12,17,18) are used to find the upper component, \( g_u(r) \), and the lower component, \( f_l(r) \), of the Dirac spinor in terms of the corrected nucleon mass \( M' \), and the parameter \( x \). Hence, the normalized spin \( \frac{1}{2} \) positive parity solution of an individual quark wave-function in the potential (6) is as follows.

\[ \Psi_{\perp\parallel}(r) = \frac{N}{\sqrt{4\pi}} \left[ \begin{array}{c}
(1-M'\sqrt{1-x^2}) \\
3\sqrt{5} \\
\end{array} \right]
\]

\[ \frac{i\sigma \cdot \vec{r}}{\sqrt{5}} \left[ \begin{array}{c}
1-x \\
1+x \\
\end{array} \right] \left[ \begin{array}{c}
(1-M'\sqrt{1-x^2}) \\
3\sqrt{5} \\
\end{array} \right]
\]

The eigenfunction (24) is got as follows:

\[ \Psi_{\perp\parallel}(r) = \frac{N}{\sqrt{4\pi}} \left[ \begin{array}{c} 
(1-yr) \\
\gamma \\
\end{array} \right] e^{-\frac{1}{2} \gamma y^2} \] ... (27)

From Eq. (27), the bag radius \( R_b \) is determined by solving the boundary equation. This shows that, the normal component of vector current vanishes at the surface of the baryon bag.

\[ \gamma^2 R_b^2 - (1+\gamma) \gamma R_b + 1 - \gamma = 0 \] ... (28)

The bag radius is determined by solving the above equation:

\[ R_b = \frac{1+\gamma - \sqrt{(1-\gamma)^2 + 4\gamma^2}}{2\gamma} \] ... (29)

The bag radius directly depends upon the quark and the nucleon masses.

3.1 Ratio of \( g_A \) to \( g_V \) for nucleon

In this Section, it is explained how, the ratio \( \frac{g_A}{g_V} \) is used as an input. The ratio of axial-vector coupling constant \( g_A \) to vector coupling constant \( g_V \) is:

\[ \frac{g_A}{g_V} = \frac{5}{3} < \sigma_z > \] ... (30)

In the relativistic case \( \frac{g_A}{g_V} \) satisfies:

\[ \frac{g_A}{g_V} = \frac{5}{3} (1-2\delta) \] ... (31)

where

\[ \delta = \frac{2}{3} \int [f_2(r)]^2 dr + \int [f_1(r)]^2 \] ... (32)

Eq. (32) contains unknown parameter \( \delta \). In order to find this parameter for the proton \((M=938\) Mev\) it can be taken that \( \frac{g_A}{g_V} = 1.255 \pm 0.006 \).

This value is measured experimentally\(^9\) or \((\delta=0.123\pm0.002)\) then get \( \delta=0.721\pm0.003 \) and from Eqs (26, 23,22) parameters of potential \( a \) and \( c \) can be found and from (21,A.6) can get \((m_q=293.139\pm0.823)\) MeV. This mass is well within the expected range.

3.2 Nucleon charge-radius

The mean-square charge radius for a hadron \( < r_{cm}^2 > \) is defined as:

\[ < r_{cm}^2 > = \Sigma e_q < r^2 >_q \] ... (33)

where

\[ < r^2 >_q = \int_{bag} r^2 \Psi_{\perp\parallel}^2(r) \Psi_{\perp\parallel}(r) \] ... (34)
Then
\[ <r_{en}^2> = \frac{\int_{0}^{\infty} r^4 |g_{+}(r)|^2 + |f_{+}(r)|^2 \, dr}{\int_{0}^{\infty} r^4 |g_{-}(r)|^2 + |f_{+}(r)|^2 \, dr} \] ...

By using the upper and lower components of the spinor, (27) and \( m=293.139\pm0.823 \) MeV from above the charge-radius of nucleons for proton it is obtained \( <r_{en}^2> = 0.802\pm0.003 \) fm. This result is closer to the observed value, 0.88\pm0.03 fm than the previous results of MIT bag model, by 10%.

### 3.3 Nucleon magnetic moment

By using the standard definitions of magnetic moment, it can be shown that, the general expression for the magnetic moment of a quark in its ground-state is as follows:
\[ \mu_q = -\frac{2}{3} e_q N^2 \int_{0}^{\infty} r^3 f_{+}(r) g_{-}(r) \, dr \] ...

From the upper and lower components of the spinor, (27) the magnetic moment of proton is obtained \( \mu_p = 2.643\pm0.012 \) fm close to 2.782 nm, the observed value. There is a considerable improvement of result of the MIT bag model (Table 1).

In Table 2, the magnetic moment results of the model is summarized and MIT bag model for several baryons. In all cases the model gives values much closer to observed values.

### 3.4 The strong coupling constant \( a_s \)

The strength of coulomb potential was calculated with regard to quark and nucleon masses. The short coulomb-like term in potential (1) is
\[ \frac{\left( \frac{b \alpha_s}{r} \right)}{r} \]
where \( b \) is the color factor, which is \(-2/3\) for the nucleon, assuming 3 flavors for quarks. Using Eqs. (7, 21 & 23), the strong coupling constant \( a_s \) is obtained.
\[ \alpha_s = \frac{3}{\sqrt{5}} \left[ \frac{1 - 3m}{M} \right]^{1/4} \] ...

In the model for the proton, where \( m=293.139\pm0.823 \) MeV, it is found that, \( \alpha_s = 0.540\pm0.002 \). This is the middle of the acceptable range. \( 0 \leq \alpha_s \leq 1 \). It should be noted that \( \alpha_s \) is an effective coupling constant in the model and is not a fundamental quantity.

### 4 Conclusion

In conclusion, valence quarks have been presented in a bag with residual interaction among them with asymptotic freedom and confinement. Hence, the results have been improved of MIT bag model, to a large extent, to get better results than many of the improved bag models. From Table 1, the static properties of proton is seen to be very close to the experiments. This method, just like MIT bag, can be applied to wide variety of hadrons and mesons.

In general, proton has been considered wherein, all cases results of the authors are much closer to experimental values than other improved bag model. For few other hadrons the authors have checked the magnetic moments which show a remarkable improvement.

Comparing Tables 1 and 2 with MIT and experiment the results show that model of the authors certainly improved the MIT bag model and hence can also prove the models.

### Acknowledgement

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### References
Appendix A: (centre-of-mass energy)

The center-of-mass energy obtained in this appendix by using Eq. (13) for each of three quarks \(i=1,2,3\). For three quarks in their lowest orbits \(k=-1\) with energy \(\varepsilon\), and mass \(m\), (13) gives:

\[
\phi_i' = \left(\varepsilon + m\right)A_i(r_i) - \left(\varepsilon^2 - m^2\right)\phi_i \quad \ldots (A.1)
\]

\[
\left[ \sum_{i=1}^{3} \frac{d^2}{dt^2} + \sum_{i=1}^{3} A_i(r_i) - 3(\varepsilon^2 - m^2) \right] \prod_{i=1}^{3} \phi_i' = 0 \quad \ldots (A.2)
\]

where \(A_i(r_i) = (\varepsilon + m) A_i(r_i)\) and introduce Jacobin coordinates. Eq. (A.2) then separates into three identical equations—one of which determines the center-of-mass:

\[
R = \frac{1}{3} (r_1 + r_2 + r_3) \quad \ldots (A.3)
\]

\[
\left[ -\frac{d^2}{dp^2} + A_i(p) - (\varepsilon^2 - m^2) \right] \phi(p) = 0 \quad \ldots (A.4)
\]

where \(p = \sqrt{3} R\)

Now, it is obvious that, the centre of mass energy is:

\[
E_{cm} = (\varepsilon^2 - m^2)^{1/2} \quad \ldots (A.5)
\]

\[
M' = M + E_{cm} = 3\varepsilon = \frac{3M}{3 - \sqrt{1-x^2}} \quad \ldots (A.6)
\]