Scaling theory and dimensional arguments for periodic solutions of spring-block model

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Earlier, the authors have studied a one-dimensional version of the Burridge-Knopoff model [Burridge R & Knopoff L, Bull Seismol Soc Am, 57 (1967) 341] of N-site chain of spring-blocks with stick-slip dynamics. Their numerical analysis and computer simulations lead to a set of different results corresponding to different boundary conditions [Gutenberg B & Richter C F, Ann Geophys, 9 (1956) 1]. The authors showed that, there are stable periodic solutions in a parameter space. They have presented elsewhere, arguments to justify the occurrence of lower edges of the window [Carlson J M & Langer J S, Phys Rev Lett, 62 (1989) 2632]. The authors follow here, the same arguments and show that the lower and upper edges of the window can be understood in terms of simple scaling arguments and can be presented as a function of two dimensionless parameters. They try to understand the origin of the results and give a theory to explain them. An improved formula for lower threshold driving velocity will be presented, which is in a good agreement with their numerical experiments inside the parameters window of periodic solutions. Also, they discuss different types of instability occurring in the system.

Keywords: Scaling theory, Spring-block model, Earthquake, Dynamical model, Plate tectonics

1 Introduction

The scaling properties of a variety of dynamical systems with power law correlations such as, earthquakes have motivated the construction of many models which can produce stick-slip behaviour. One simple model with complex behaviour that has received considerable attention is the spring-blocks model due to Burridge and Knopoff1. They introduced a dynamical model of plate tectonics in which, an elastically-coupled array of slider blocks are represented between two plates. The blocks are coupled elastically, to one plate and coupled to the other plate via a stick-slip friction function. The elastically-coupled plate moves with constant velocity \( V \) and the other plate is fixed. They carried out some experiments and simulations on very small systems and their results showed that, their model displayed the power-law, consistent with the Gutenberg-Richter law1. According to empirical Gutenberg-Richter law, the distribution of earthquake events is scale-free over many orders of magnitude in energy.

Carlson and Langer discussed2-3 this model for any blocks in which, the friction function for a given slider is a velocity-weakening type. They used free boundary conditions and demonstrated that, this homogeneous model had chaotic solutions in a low velocity regime. The version of the model used by the authors was originally presented by Carlson and Langer. The authors demonstrated that, this homogeneous model with free boundary conditions had stable, nearly periodic solutions in a certain window of parameter space1. This is by contrast with the result of Carlson and Langer who found that, in their region of parameter space, the only stable solutions were chaotic. The model is embodied in the following equations:

\[
\frac{d^2 X_j}{dt^2} = k_e (X_{j+1} - 2X_j + X_{j-1}) + k_p + F(X_j)
\]

\( j=1,2,\ldots, N; X_0 = X, X_{N+1} = X_N \)

...(1)

Here, \( X_j \) is a position of slider located at site \( j \), and only nearest neighbour interactions is considered in the system. Also, \( \dot{X}_j \) represents the velocity of slider \( j \). The conditions in Eq. (1) represent the end effects of the system and, using the free boundary
conditions, can be written as the equations of motion for the first and last particles:

\[ m \frac{d^2 X_i}{dt^2} = k_i (X_2 - X_i) + k_p (vt - X_i) + F(X_i) \quad \ldots (2) \]

\[ m \frac{d^2 X_N}{dt^2} = k_i (-X_N + X_{N-1}) + k_p (vt - X_N) + F(X_N) \quad \ldots (3) \]

In the force Eq. (1), \( F(X) \) is the non-linear velocity-dependent friction force, and in a relevant choice, to confirm the decreasing form of friction function, takes the form given below:

\[ F(X_j) = -\frac{F_0}{1 + \left| \frac{\dot{X}_j}{V_i} \right|} \quad \text{sgn} \left( \dot{X}_j \right), \quad \dot{X}_j \neq 0 \quad \ldots (4) \]

where, \( V_i \) is a reference velocity which characterizes the velocity-dependent of the friction. \( V_i = 1 \) can be taken by selecting the proper units, as it was done in previous work. Threshold value of friction function, \( F_0 \), is the quantity measuring the static friction function to be overcome before a shock or slipping takes place. The non-linear stick-slip friction function \( F(\dot{X}_j) \) has the following effect:

1. It balances the sum of the elastic terms, if the modulus of the sum is less than \( F_0 \), and the velocity \( X_i \) is zero.
2. If the sum lies outside this range and the velocity \( \dot{X}_j \) is zero then it is reduced by \( \pm F_0 \).
3. Finally, if the velocity \( \dot{X}_j \) of the slider is non-zero, then the friction function is given by Eq. (4).

2 Results and Discussion

The first step, in order to have some insight into behaviour of the system is to find a single quantity, which captures this behaviour. The average friction force is a good magnitude, as it was shown in Refs 5,6. Another important quantity which the authors are interested and was shown in their previous work is, the time series of the total elastic force, acting on the chain, \( P(t) \). This is the sum of all terms proportional to \( k_i \) in Eq. (1). The terms proportional to \( k_i \), as an internal force, add up to zero.

The authors note here that, the model contains five parameters viz. \( m, k_i, k_p, F_0 \), and \( V_i \) and also \( N \) the number of sliders in the system which, in the case of free boundary conditions, is not an interesting parameter. It also contains the control parameter \( v \) which is the velocity of the upper plate, relative to the lower plate. In their previous work the authors showed that, for each point in some region of values of the five parameters, there is an interval in \( v \) and in \( F_0 \) in which the solutions are quiet, stable and nearly periodic. This is illustrated in Fig. 1 where the force trace is presented for three values of

![Time series of total elastic force](image-url)

Fig. 1 — The force trace \( P(t) \) plotted against time \( t \) for three values of \( v \), the first below \( v_i \), the second between \( v_i \) and \( v_{11} \), and the third above \( v_{11} \). The parameter values are \( N=1000, F_0=20, k_i=40, k_p=50 \), and \( V_i=1 \).
For the intermediate value of $v$, the trace is quiet and nearly periodic but, the trace is noisy and chaotic for other values. It is obviously important to discuss the shape of this window of periodicity, because the model appears to describe a wide range of dynamical ordering phenomena discussed in an earlier paper and elsewhere. In other words, it is important to describe the velocity at the upper threshold $v$, and the velocity at the lower threshold $v$, as functions of the five parameters which define the model. It is quite difficult to present or even list a function of such a large number of parameters.

In this paper, the authors present calculations of the functions $v>v$ and $v<v$ for a limited range of values and then show that, all the information can be collapsed onto a function of two dimensionless variables, by using simple scaling arguments based on dimensional analysis. In other words, they have a tractable amount of information to present and can begin thinking about a theory to explain the results which, thus far, are purely numerical, apart from the scaling argument.

In the window of quiet behaviour, the chain supports the propagation of dispersion-less waves in a certain range of frequencies which are generated by the oscillations of the end of the chain. If the frequency of oscillation of the ends is in this range, then quiet behaviour follows but, if the frequency is out of this range, then noisy behaviour follows as illustrated by two of the curves of Fig. 1.

The upper threshold $v_u$ is determined by this condition and this can easily be checked by constraining the ends of the chain to oscillate with whatever frequency and amplitude one wishes. It is found that the upper threshold can be pushed to arbitrary high values by choosing the frequency and amplitude, appropriately. From these appropriate values of frequency and amplitude onwards, it seems that, the system is under the control of forced frequency, so it shows the behaviour with beat modes and frequency, compatible with external frequency. Then, for higher values of $v$, every mass in the system is always moving, so we cannot observe the stick-slip phenomenon. The lower
threshold $v_1$ cannot be reduced, very significantly, by this technique.

This can be seen in Fig. 2 where it is shown that, choosing the frequency of oscillation of the ends of the chain with a selected initial conditions and given parameters in the caption reduces $v_1$ modestly from 0.58, which is the value obtained when the ends are free, to only 0.34 and no further. Also, values of $v_1$ greater than 0.58 are not found and this value corresponds to the upper threshold for frequency; i.e., $\omega_1 = 10.0$. In this figure, the values of $v_1$ and $v_L$ are compared for the same values of forced frequency. For values of $v$ below the reduced estimate of $v_1$ obtained by forcing the ends to oscillate with a definite frequency, dispersion-less waves cannot propagate down the chain and the chaotic behaviour emerges. This point actually corresponds to $\omega_1 = 5.0$, which is the lower threshold for external frequency. This is also the value at which the two threshold velocities coincide with each other. For values of frequency $\omega$ greater than 10.0 and values of $v_1$ greater than 0.95, the system is totally under the control of forced $\omega$ and so the Fourier transform of the force trace shows a very distinct sharp peak corresponding to the value of the applied frequency. The lower threshold is, therefore, more interesting; it is an intrinsic property of the chain whereas the upper threshold is related to the interaction of the chain with its ends. Here, for the sake of quick and easy calculations estimates of $v_1$ and $v_L$ are obtained from a chain with free ends.

The results of the numerical calculations of the lower threshold as function of the four parameters $F_0, V_1, k_1$, and $k_p$ normalised to $m = 1$ have been reported in an earlier paper\(^1\). As it was shown, there was no apparent to the dependence of $v_1$ on the parameters. However, the authors could easily find a heuristic argument which produced the correct trend. Firstly, as expected in earlier works\(^8,10\) the wave consists of a series of pulses of rapidly moving sliders between fixed sliders. Thus, for most of the sliders at any instant, the frictional force given by Eq. (4) is either zero or proportional to the product $F_0 V_1$, because we can ignore the constant term in the denominator. Thus, $v_1$ will depend on these two parameters through this combination only. The only error in this argument is, due to its misrepresentation of the frictional threshold effects. Interactions in the chain which lead to the wave-like character of the solution are due to the product $F_0 V_1 / k_p$ divided by a characteristic time. The time can be guessed at, by considering the first term on the right of Eq. (1) and this gives $\sqrt{m / k_1}$. Thus:

$$v_1^2 = \frac{F_0 V_1 \sqrt{k_1}}{k_p \sqrt{m}} \quad \ldots(5)$$

There is clearly some truth in this expression since, it produces the correct trends viz. increasing $v_1$ with $F_0, V_1$ and $k_1$ and decreasing $v_1$ with $k_p$ but the detailed dependence is not accurate.

Here, the authors also present the numerical calculations for the upper threshold $v_u$ as function of the same parameters $F_0, V_1, k_1$, and $k_p$. It is observed, $v_u$ behaves slightly different from $v_1$; e.g., the behaviour of $v_u$ with respect to $k_p$ is decreasing rather than increasing. Also, the dependence of $v_u$ on $k_1$ is similar to that of $v_1$ except for the peak of low values of $k_p$. The shape of the dependence of $v_u$ on $F_0$ and $V_1$ which has a kink suggests that, two distinct physical mechanisms govern the upper threshold. An important result is seen where, $v_1$ and $v_u$ are plotted against $k_p$. For very small values of $k_p$ (i.e., at the far left of the diagram) the ordering is established for a range of values of $v$ but only after very long waiting times. The reason for this is that, the system is behaving almost like $N$ independent oscillators and the correlations take a long time to develop because of the very weak coupling between the oscillators. Furthermore, the spatial behaviour is more ordered than temporal behaviour. The broad window around $k_p = 0$ has a very important feature. This pattern can be seen by calculating the Fourier transform.

The rough agreement with the heuristic relation (5) gives us confidence to produce a scaling argument based on simple dimensional analysis. There is one parameter $V_1$ and one combination of parameters $F_0/(m k_p)$ which have the units of velocity and there are two dimensionless parameters.

$$\alpha = k_1 / k_p \quad \text{and} \quad \beta = V_1 / V_0$$

In the paper by Vieira et al.\(^7\), four parameters with the units of velocity are listed, but one is the control parameter $v$ and another is a combination of
parameter $V_1$ of the authors and the dimensionless parameter $\alpha$. This suggests that we can write:

$$v_1 = V_1 F(\alpha, \beta)$$  \hspace{1cm} (7)

where, $F(\alpha, \beta)$ is a dimensionless function of two dimensionless variables. Accordingly, the authors have plotted in Fig. 3 the ratio $v_1 = v/V_0$ against $\alpha$ for fixed $\beta$. Also, Fig. 4 shows the behaviour of $v_1 = v/V_0$ and $v_\ast = v/V_0$ in terms of $\beta$ for fixed $\alpha$. These plots contain all the information obtained before, in terms of $k, k', F_0$ and $V_1$. It can be seen that, all the data collapses onto two curves thus verifying dimensional analysis of the authors and displaying the data in a reasonably compact form. The authors are confident that, any other calculation will give a point on the two-dimensional surface

![Figure 3](image1.png)

**Fig. 3** — The lower and upper threshold velocities $v_1^\ast$ and $v_\ast$ plotted against $\beta = V_1/V_0$ for $\alpha = 0.8$ and different values of other parameters

![Figure 4](image2.png)

**Fig. 4** — The force trace $P(t)$ plotted against time $t$ for different values of driving velocity and friction threshold. First, for a value of $v=0.65$ and $F_0=20$, inside the parameter window for which the ordered periodic solution emerges (point A). Second, for a value of $v=0.9$ and $F_0=30$ outside the parameter window of the periodic solution (point B). Finally, for a larger value of velocity, $v=1.5$ and $F_0=25$. Here, the waves driven in from the ends become unstable (point C). The other parameter values are $N=200$, $k' = 40$, $k = 50$, and $V_1 = 1$.
Fig. 5 — The configuration of the chain after a long time for different values of driving velocity and friction threshold. First, for a value of $v=0.65$ and $F_0=20$ inside the parameter window for which the ordered periodic solution emerges (point A). Second, for a value of $v=0.9$ and $F_0=30$ outside the parameter window of the periodic solution (point B). Here, the instability emerging from the centre and the point of collision becomes unstable. Finally, for a larger value of velocity, $v=1.5$ and $F_0=25$. Here, the waves driven in from the ends become unstable (point C). The other parameter values are $N=200$, $k_c=40$, $k_p=50$, and $v_f=1$. Lines are a guide to the eye and longitudinal displacements are plotted laterally for clarity of presentation.

Fig. 6 — The phase diagram in the region of the upper threshold. The region 1 is the stable region below the threshold. The region 2 is the region when the waves driven in from the ends become unstable. The region 3 is the region where the centre becomes unstable and wobbles about. The dashed line is the extension of $L_1$. There is no extension of $L_2$ because to observe instabilities of the centre the waves driven in from the ends must first become unstable.

described by their two curves! It remains to understand the shape of the universal function of two parameters (presented in Figs 3 and 4) that emerges from their calculations and dimensional analysis.

In a series of calculations, the authors find the dependence of $F(\alpha, \beta)$ on these two parameters. First of all, in Fig. 5 the authors consider the relation $(k_p, v)$. The authors find that, a fit $v \sim k_p^{-3/4}$ to the data is good. Then we see that except in the region of very low values of $\alpha$ (which corresponds to the anomalous region of weakly-coupled oscillators with very small values of $k_c$, $v_1$ varies as $\alpha^{1/2}$ when $\beta$ is fixed. This can be seen clearly from Fig. 3. Fig. 4 demonstrates that, a power-law fit of the form $v_1$...
except for very small and very large values of $\beta$. So, the result (5) is improved by using facts as follows:

$$v_1^2 \approx \alpha^{1/2} \beta^{1/2}$$  \hspace{1cm} (8)

$$v_2^2 \approx V_0^2 \frac{k_p}{k_p^{3/2}}$$  \hspace{1cm} (9)

To obtain the value of the coefficient of proportionality, several calculations are carried out on system and the results compared with each other. Then it is found:

$$v_1 = V_0 \frac{C \alpha^{1/2} \beta^{1/2}}{k_p^{3/4} m^{1/4}}$$  \hspace{1cm} (10)

where, $C=0.28\pm0.01$. This obviously opens up the possibilities for further work. Also, it can be seen that, the relation $v_2^2 \approx C^{1/2}$ is a good fit to the data for upper threshold velocity, except for very low values of $\alpha$. Again, by using a heuristic argument a trend of the behaviour of $v_2$ can be found, but as said earlier, the upper threshold is related to the interaction of the chain with its ends and can be controlled by whatever frequency is applied to the system. Also, the results on $v_2$ show that, there is a kink in the behaviour of upper threshold.

It seems that, the two portions of the upper threshold curve are due to different physical mechanisms. To investigate in more depth, the authors consider the behaviour of $(F_0, v)$ curve (Fig. 6). This graph tells more about the types of instability occurring in the system. The authors study in an accurate way on two sides of the singular point. For the value $F_0=20$, say, and for values of $v$ under the threshold (say, $v=0.65$, point A in Fig. 6), for a value of driving velocity just over the upper threshold (for example, $v=0.73$) interesting effects are observed and instabilities seem to propagate in from the ends. The other branch of the curve is now investigated (for the value say, $F_0=30$ and several values of the driving velocity below and above the threshold). When $v=0.65$ (below the threshold) there is total synchronization and there are stable inward waves driving from the ends, although it takes a long time for the stability to emerge and for the centre of collision to reach stability. When $v=0.9$ (above the threshold, point B in Fig. 6) the centre becomes unstable and wobbles around. The waves driven by the ends are still stable. In other words, the point of collision becomes unstable and wanders about and becomes ill-defined; the region in which the instability occurred is narrow and depends on the value of $v$ and $F_0$. Basically, instability to middle does not propagate and remains localised. However, the middle wanders a bit (see Figs 4, 5, and 6). When $v=1.5$ (point C in Fig. 6) which is in a yet more unstable region of the phase diagram, the waves driven in from the ends become unstable region (see Figs 4, 5, and 6). Fig. 6 gives us the whole information. This is the phase diagram in the region of upper threshold velocity. The authors recognize three different regions corresponding to different behaviours: The region 1 is unstable and wobbles about. The dashed line is the extension of $L_v$. There is no extension of $L_v$, because to observe instabilities of the centre, the waves driven in from the ends must first become unstable and this does not obviously happen for this case.

4 Conclusion

Finally, it is concluded that, a simple argument based on dimensional analysis enables us to present a wide collection of data in a reasonably compact form. The data is of some importance, because it relates to pattern formation in a widely applicable non-linear dynamical system viz. the Burridge-Knopoff model. The authors have presented an improved relation for the lower threshold velocity which confirms their numerical results inside the parameter window for which the periodic solutions emerge. The kink in $v_1$ needs to be explored in an accurate way and it might open up more possibilities for further work. It is realized that, there are three types of instabilities: (i) For small values of the friction in the upper threshold region, the ends do not drive a stable wave into the chain; i.e., the interaction of the ends with the bulk is responsible for the instability. The instability can spread through the whole system. (ii) For larger values of the friction and the driving velocity, the point of collision becomes unstable, but this instability remains localised around the centre. (iii) The wave itself becomes unstable; i.e., the chain cannot support a wave-like solution and this type of instability usually happens for the lower threshold of driving velocity.

There are a number of important, unresolved questions about the behaviour of the model, which
have enormous implication for any type of eventual theoretical understanding. The numerical study and analysis of the authors tries to address those issues and answer those questions.

References