Numerical computation of the interaction of surface waves with submarine pit breakwaters

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Present study consists the interaction of surface waves with submarine pit breakwaters. A vertically-integrated nonlinear dispersive wave model with inclusion of a semi-empirical wave breaking effect, in non-orthogonal curvilinear coordinate system is used for simulating waves. The staggered grid system is used. Matrix equation of the finite difference method is solved through an iterative approach at each time step. Numerical method results in implicit formulations for the momentum equations and explicit formulation for the continuity equation. The numerical model has been validated through comparisons of computational results with experimental measurements of the wave propagation over an asymmetrical trapezoidal bar, over shoals and the regular wave diffraction results for rectangular pit. Simulations show an acceptable level of agreement and give confidence for practical applications.

Keywords: Boundary fitted coordinates; Submarine pit; Wave diffraction; Nonlinear waves.

Introduction

Breakwaters are structures constructed to provide shelters in coastal zones. They may be built above the water surface or submerged in the water. Submerged ones are called submarine breakwaters, and they provide protection through the scattering and refraction of the incident wave energy. Submarine breakwaters have been studied numerically by several researchers as outlined in Ouahsine et al\textsuperscript{1} and Hur et al\textsuperscript{2}. The diffraction of waves by a rectangular submarine pit has been studied by several researchers\textsuperscript{3-14}. Some of these studies were based on 2D models; others were based on shallow water theory. Furthermore, most of the work done was restricted to regular waves.

In the present work, the nonlinear wave model of Nadaoka et al\textsuperscript{15} with the inclusion of a semi-empirical wave breaking effect is transformed into a boundary fitted non-orthogonal curvilinear coordinate system in the terms of the contravariant velocity components. The computer code developed for the numerical solution of the transformed equations first computes the metrics and jacobians for the grid structure, and the numerical method then uses finite difference approximations on staggered grids, obtaining the horizontal velocity components from the momentum equations implicitly and the free surface elevation from the continuity equation by an explicit formulation. The proposed modified wave model is capable of modelling the combined effects of nonlinear refraction-diffraction problem for the existence of the pits in the domain. Although the examples presented here do not require the use of boundary fitted coordinates, this work aims to give a more general formulation which can be applied.

The present numerical model has been validated against wave propagation over an asymmetrical trapezoidal bar by Beji and Battjes\textsuperscript{16}, over a circular shoal by Ito and Tanimoto\textsuperscript{17}, over a complex topography by Berkoff et al\textsuperscript{18} and the regular wave diffraction results of Williams\textsuperscript{10}. The numerical simulations presented here show that, the present numerical model give confidence for calculating the effect of the various wave and pit parameters on the diffraction characteristics of pit systems.

Materials and Methods

Breaking of waves

Within the framework of Boussinesq equations wave models, many wave breaking formulas have been suggested: Madsen et al\textsuperscript{19}, Kennedy et al\textsuperscript{20}, to name a few studies. Simplest of those adds an eddy-viscosity term to the momentum equation, while a more sophisticated approach adopts a surface roller approximation. Proposed modified wave model
predicts dissipation of energy due to breaking in an empirical manner as a numerical dissipation which is related to the mean velocity gradient that increases with wave steepness. Thus an energy dissipation model is built into the equation by the addition of a wave breaking damping term \( \alpha B \), to simulate the wave breaking effect. Magnitude of the coefficient \( \alpha \) is infinitesimally small when breaking does not occur, but grows to a significant value when wave steepness becomes large and breaking starts. Then the top of the wave crest is peeled off as the damping term becomes effective. The breaking term is considered in the form of an eddy viscosity term as \( \alpha \nabla^2 \mathbf{u} \) and then expressed in the curvilinear coordinates as

\[
\mathbf{B} = \mathbf{V} \cdot \nabla^2 \mathbf{u}
\]

which reads in the curvilinear coordinates as

\[
B_\xi = B_{\eta \eta} = \frac{q_\xi^2}{C_p^2 \eta} \left[ \left( \frac{C_p^2}{g} + \zeta \right) U_\eta + \frac{C_p^2}{g} \eta \xi \right] + \frac{q_\eta^2}{C_p^2 \eta} \left[ \left( \frac{C_p^2}{g} + \zeta \right) U_\xi \right] + \zeta + \frac{q_\xi^2}{C_p^2 \eta} \left[ \xi \eta + \frac{\xi}{C_p^2} \right] + \frac{q_\eta^2}{C_p^2 \eta} \left[ \xi \eta + \frac{\eta}{C_p^2} \right]
\]

\[
\ldots (1)
\]

where \( \alpha \) is a breaking coefficient signifying the intensity of breaking, and explained in the following section. Note that for the special case of unidirectional waves only the terms containing \( U_\eta \) remain.

**Governing equations**

The wave model used in this work is the one-component form of the fully-dispersive vertically-integrated nonlinear model of Nadaoka et al\(^{15}\) as expressed in non-orthogonal curvilinear coordinates by Beji and Barlas\(^{21}\) with insertion of a semi-empirical wave breaking effect. Model is valid for both shallow and deep water waves. It can reproduce the cnoidal and second order Stokes waves well, besides simulating narrow banded nonlinear random waves over arbitrary depths (see Nadaoka et al\(^{15}\) for details).

Regardless of the shape of the physical domain, wave equations are solved in a rectangular computational domain using finite difference approximations. Fig. 1 shows a sketch of an arbitrary physical domain and the corresponding computational domain.

In non-orthogonal coordinate system, the \( \xi \)-coordinate is aligned with the incident wave direction. Momentum equations reflect the momentum balance in the directions of the curvilinear coordinate components \( \xi \) and \( \eta \). This requires the appropriate addition of the Cartesian component momentum equations after transformation into the \( (\xi, \eta, \zeta) \) coordinate system. Momentum equations in the \( \xi \) and \( \eta \) directions are obtained from the transformed \( x \) and \( y \) momentum equations in the following way:

\[
\xi_\xi (x\text{-momentum}) + \xi_\eta (y\text{-momentum}) = 0
\]

\[
\eta_\xi (x\text{-momentum}) + \eta_\eta (y\text{-momentum}) = 0 \quad \ldots (2)
\]

The transformed forms of the wave equations in non-orthogonal curvilinear coordinates \( (\xi, \eta, \zeta) \) are given by,

\[
\xi_\xi + J \left( \frac{C_p^2}{g} + \zeta \right) U_\eta + J \left( \frac{C_p^2}{g} + \zeta \right) Y_\eta = 0 \quad \ldots (3)
\]

for the continuity,

\[
r U_\xi \xi_\xi \xi + \xi_\eta Q_\xi + \xi_\xi Q_\eta = \frac{q_\xi^2}{C_p^2 \eta} \left[ \left( \frac{C_p^2}{g} + \zeta \right) U_\eta + \frac{C_p^2}{g} \eta \xi \right] + \frac{q_\eta^2}{C_p^2 \eta} \left[ \left( \frac{C_p^2}{g} + \zeta \right) U_\xi \right] + \zeta + \frac{q_\xi^2}{C_p^2 \eta} \left[ \xi \eta + \frac{\xi}{C_p^2} \right] + \frac{q_\eta^2}{C_p^2 \eta} \left[ \xi \eta + \frac{\eta}{C_p^2} \right] + \alpha B \quad \ldots (4)
\]

for the \( \xi \)-momentum,

\[
r V_\eta \eta_\zeta + \eta_\xi Q_\eta + \eta_\chi Q_\xi = \frac{q_\xi^2}{C_p^2 \eta} \left[ \left( \frac{C_p^2}{g} + \zeta \right) U_\eta + \frac{C_p^2}{g} \eta \xi \right] + \frac{q_\eta^2}{C_p^2 \eta} \left[ \left( \frac{C_p^2}{g} + \zeta \right) U_\xi \right] + \zeta + \frac{q_\xi^2}{C_p^2 \eta} \left[ \xi \eta + \frac{\xi}{C_p^2} \right] + \frac{q_\eta^2}{C_p^2 \eta} \left[ \xi \eta + \frac{\eta}{C_p^2} \right] + \alpha B
\]

for the \( \eta \)-momentum equations with,

\[
Q = g\zeta + \frac{1}{2} \left( \frac{C_p^2}{g} C_\eta / C_p \right)^2 \left( \eta \xi U_\eta + \xi \eta U_\xi - 2 \zeta \xi \eta U_\xi \right) \quad \ldots (6)
\]

\[
\xi_\chi^2 = \xi_\xi^2 + \xi_\eta^2, \quad \eta_\chi^2 = \eta_\xi^2 + \eta_\eta^2, \quad \xi \eta_* = \xi \eta + \xi \eta_*, \quad C_* = C_p^2 (1 - r) J \quad \ldots (7)
\]

\[
J = \frac{1}{x_\eta y_\eta - x_\xi y_\xi} \quad \ldots (8)
\]

\[
\alpha = \sqrt{\mu e^{\beta \zeta}} \quad \ldots (9)
\]

where, \( r = C_g / C_p \), \( g \) is the gravitational acceleration, \( \zeta \) is the free surface elevation. \( C_p \), and \( C_g \) denote respectively the phase and group velocities, computed according to the linear theory for a prescribed dominant frequency \( \omega \) and a given local depth \( h \).
The breaking term $B$ is defined in Equation (1). $\mu$ is the eddy viscosity, which is taken $10^{-4}$. The coefficient $\beta$ is taken as 18.4 for all the computations. $J$ is the Jacobian of the transformation. $\xi_x, \xi_y, \eta_x, \eta_y$ are the metrics of the grid system. Subscripts indicate partial differentiation with respect to the indicated variable in the curvilinear coordinates. $U^*$ and $V^*$ are defined as,

$$U^* = u y_\eta - v x_\eta = \frac{u \xi_x + v \xi_y}{J} \quad \ldots (10)$$

$$V^* = -u y_\xi - v x_\xi = \frac{u \eta_x + v \eta_y}{J} \quad \ldots (11)$$

Above wave equation in curvilinear coordinates can simulate linear and nonlinear waves over arbitrary depths (shallow and deep) and arbitrary lateral boundaries surrounding the domain.

**Boundary conditions**

Three different boundary conditions are used: incoming boundary condition, symmetry condition and outgoing boundary condition. At the incoming boundary, which is taken along the $\eta$-axis at $\xi=0$ (or $\xi=n$), the free surface displacement $\zeta$ is specified by assigning time series of a definite waveform. For unidirectional waves, $U^*$ is taken

$$U^* = \frac{C \cdot \xi \cdot \xi_x}{\left( \frac{C^2}{g} + \xi \right) J} \quad \ldots (12)$$

The symmetry condition can be specified as;

$$U^* = U^*, \quad V^* = -V^* \quad \ldots (13)$$

The radiation condition for outgoing waves uses Sommerfeld’s equation. Sommerfeld’s equations in curvilinear coordinates in the $\xi$-direction can be written as

$$\left[ (\xi_x \eta_y - \xi_y \eta_x) U_\xi^* + C \left[ \xi_y \eta_x - \xi_x \eta_y \right] U_\eta^* + \eta_x \eta_y \left( \xi_x - \xi_y \right) U_{\xi \eta}^* \right]$$

$$+ C \xi_x \xi_y \left[ \xi_x \eta_y - \xi_y \eta_x \right] V_\xi^* + \left( \eta_y - \eta_x \right) V_{\xi \eta}^* = 0 \quad \ldots (14)$$

**Numerical approach**

The numerical algorithm employed in this study is based on finite difference approximations, previously developed by Beji and Barlas. A brief summary is presented in the remainder of this section.

Solutions of Equations (3)-(5) may be accomplished using finite difference approximations. Fig. 2 shows the staggered grid orientation for the variables in the physical domain. Computer code developed for the numerical solution of the transformed equations first establishes the physical grid structure and then computes the metrics and jacobians at the grid points. Momentum equations are solved implicitly and then the continuity equation is solved explicitly. Second order finite difference formulae are used in the partial derivatives of the space and grid metrics terms. Two time levels are used and all the derivatives are centered at the mid-time level $t+\Delta t/2$, $t$ being the current time and $\Delta t$ the time step. The incident wave propagation direction is taken along the positive $\xi$-axis. To obtain the velocity field, $\xi$-momentum equation is solved first to obtain $U^*$ for the new time level $t+\Delta t$, assuming the new time level values of $\zeta$ and $V^*$ are known. $\eta$-momentum equation is solved later for $V^*$ using $U^*$ as computed from the previous step. The free surface displacement $\zeta$ is obtained from the continuity equation. In the computational domain $\Delta \xi$ and $\Delta \eta$ is taken 1. Simulations are performed for making the Courant number in the wave propagation direction $C_r = C \Delta t / \Delta x$ equal to one. As the computational domain is not damped, the computation is ended before the reflected waves reach the incoming boundary.

All the computations were performed on a Pentium i7 2.8 GHz personal computer, with 8 GB of RAM. Since the computations involve certain approximations, an iterative procedure is required; during the computations the relative error cut is $10^{-5}$, i.e. the
difference between the iteratively computed values of the free surface displacement $\zeta$ at every grid point should be less than $10^{-5}$. A cumulative maximum error (%) decrease during the computations is given in Fig. 3.

**Results and Discussion**

In order to verify the reliability of the numerical approach and the accuracy of the computer code, known cases of directional wave propagation simulations are given first. Waves over an asymmetrical trapezoidal bar, waves over a circular shoal and waves over an elliptic shoal on a sloping beach cases are standard test cases for nonlinear refraction-diffraction problems. Computations for submarine pits are presented later.

*Breaking waves over an asymmetrical trapezoidal bar*

Data from the experiments of breaking wave propagation over an asymmetrical trapezoidal bar carried out by Beji and Battjes\(^\text{16}\) is used for testing the numerical model. Bathymetry and locations of the sensors is depicted in Fig. 4. JONSWAP type random incident waves with peak frequency $f_p=0.5$ Hz, significant wave height $H_s=1.8$ cm are specified at the incoming boundary. $\Delta t$ equal to $1/100$ of the initial wave period, $\Delta x$ approximately equal to $1/70$ of initial wavelength, and 1400 grid points are used. In Fig. 5 time histories of the measured and computed data are compared for the stations 2-7. Although the model dissipation of very steep waves is higher than the measurements, overall the simulation of spilling random breaking waves by the model appears to be acceptable.

*Waves over a circular shoal*

Wave propagation over a circular shoal was first measured by Ito and Tanimoto\(^\text{17}\). The wave tank used in the experiments was 3.2 m long and 2.4 m wide. Circular shoal centered 1.2 m away from the entrance. Bathymetry is given as

$$h(x,y) = \begin{cases} h_m + e_0 r^2 & \text{for } r < R \\ h_0 & \text{for } r \geq R \end{cases} \quad \ldots (15)$$

Where

$$r^2 = (x-x_m)^2 + (y-y_m)^2 \quad e_0 = \left(\frac{h_0-h_m}{R}\right) \quad \ldots (16)$$

in which $h_0$ is the water depth outside the shoal, $h_m$ is the water depth at the shallowest point over the center of the circular shoal, $R$ is the radius of the circular shoal. Fig. 6 shows the bathymetry according to these definitions.

The incident wave period is $T=0.5$ s, the wave length $L=0.38$ m, $k=16.34$ rad/m, the amplitude is $a=0.0052$ cm, $h_0/R=0.1875$, $h_m/R=0.0625$, and $L/R=0.5$. The $\Delta t=T/40$ s, $\Delta x=L/43$ m and 36 0× 22 grid system is used. Sinusoidal waves with uniform amplitude are imposed at the incoming boundary, while Equation (14) is used as the radiation condition at the outgoing boundary. Fig. 7 shows a perspective view of a fully developed wave field in presence of circular shoal. In Fig. 8 numerically computed wave amplitude variations are compared with the measured experimental data along the centerline. Wave amplitude is normalized using $\zeta/a$. The numerically computed results are in good agreement with the experimental data.

*Waves over an elliptic shoal on a sloping beach*

This case tests the model for waves propagating over a complex topography; the experiment reported by Berkoff et al\(^\text{18}\). Wave tank used in the experiments has a bathymetry consisting of an elliptical shoal situating on a plane beach with a slope of 1/50. Slope starts from a region of constant depth of $h_0=0.45$ m and the slope were rotated clockwise at an angle of 20° as shown in Fig. 9. Region was 20 m long and
20 m wide. Outside the shoal region the water depth may be represented by,

$$h(x, y) = \begin{cases} 
  h_0 & \text{for } G < -5.84 \\
  h_0 - 0.02(5.84 + G) & \text{for } G \geq -5.84
\end{cases} \quad \ldots (17)$$

Where

$$G = (x - 10.5)\cos 20^\circ - (y - 10)\sin 20^\circ \quad \ldots (18)$$

$$H = (x - 10.5)\sin 20^\circ + (y - 10)\cos 20^\circ \quad \ldots (19)$$

The shoal boundary is given by,

$$\left(\frac{G}{3}\right)^2 + \left(\frac{H}{4}\right)^2 = 1 \quad \ldots (20)$$

in the shoal region the water depth may be represented by

$$h(x, y) = h(x, y) + 0.3 - \sqrt{1 - \left(\frac{G}{3.75}\right)^2 - \left(\frac{H}{5}\right)^2} \quad \ldots (21)$$

The incident wave period is $T=1$ s, the wave length $L=1.5$ m, $k=4.21$ rad/m and the amplitude is $a=2.32$ cm. The $\Delta t=T/40$ s, $\Delta x=L/41$ m and $540 \times 50$ grid system is used. Sinusoidal waves with uniform amplitude are imposed across the entrance, while Equation (14) is used as the radiation condition at the end of the computational domain. Fig. 10 depicts a perspective view and Fig. 11 shows contour plot of the fully developed wave field over the elliptic shoal on a sloping beach. In Fig. 12 numerically computed normalized wave amplitude variations are compared
with the measured experimental data along the centerline for waves propagating over the elliptic shoal. As the figure reveals the asymmetrical depth bathymetry causes unwanted reflections both from the outgoing boundary and from the side walls, hence the level of agreement with the experiment is good even for such a case involving complicated reflection-diffraction example.

Single square pit breakwater

The waves propagating over a single square pit case \((w_1=w_2)\) as shown in Fig. 13 was first obtained numerically by Williams\(^{10}\). Results are presented in terms of dimensionless pit width, length and depth parameters \(w_1/L, w_2/L\) and \(d/h\), where \(w_1, w_2\) and \(d\) denote the single pit width, length and depth respectively. \(L\) is the incident wave length. Fig. 14 shows the diffraction pattern for the case of a single pit with, \(w_1/L=1, w_2/L=1\) and \(d/h=3\). Incident wave amplitude is taken as \(a=0.1\) m, which gives \(a/h=0.067\).
and may be considering mildly nonlinear at the incoming boundary. Dimensionless wave frequency $kh=0.167$, and $T=15$ s. Contour lines indicate the values of the diffraction coefficient. It can be seen from the Fig. 15 and Fig. 16 that in lateral and front sides of the pit a partial standing wave system develops, while in the lee of the pit a reduced wave height shadow zone exists. Fig. 16 depicts a perspective view of the fully developed wave field over the single pit case.

Single rectangular pit breakwater

To analyze the sensitivity of the wave amplitude estimates to the various wave and pit parameters in regular waves, the pit dimensions and pit depth were systematically varied. Geometry for the rectangular pit configuration is defined in Fig. 13. The numerical results for the case of a single pit for the minimum and maximum diffraction coefficients in both the exterior and interior regions are presented in Fig. 17–22. In Fig. 17 dimensionless pit depth $d/h$ ranges from 1.33 to 3. This figure presents the results for the maximum and minimum diffraction coefficients as a function of pit depth, as measured by the relative water depth $d/h$ for regular wave case. As can be seen from the figure, the value of $d/h$ increases, the level of protection provided by the pit also increases. However, it appears that the maximum diffraction coefficient increases as much as the minimum diffraction coefficient decreases. In McDougal et al.¹², they calculate the minimum diffraction coefficient appears to be higher. In particular, Fig. 18-20 present the results for the

![Image](https://example.com/image1.png)

Fig. 14—Contour plot of diffraction coefficient for single pit breakwater with $w_1/L=1$, $w_2/L=1$, $d/h=3$ and $kh=0.167$

![Image](https://example.com/image2.png)

Fig. 15—Surface projection of diffraction coefficient for a single pit with $w_1/L=1$, $w_2/L=1$, $d/h=3$ and $kh=0.167$

![Image](https://example.com/image3.png)

Fig. 16—Perspective view of fully developed wave field over a single pit with $w_1/L=1$, $w_2/L=1$, $d/h=3$ and $kh=0.167$

![Image](https://example.com/image4.png)

Fig. 17—Variation of maximum and minimum diffraction coefficient with dimensionless pit length for $w_1/L=0.5$, $w_2/L=1$, $kh=0.167$. Notations: minimum diffraction coefficient (ooo), maximum diffraction coefficient (•••).
diffraction coefficients when the pit length, \( w_1 \), and width, \( w_2 \), are gradually increased. \( w_1/L \) varies between 0.1 to 1, and \( w_2/L \) varies between 0.5 to 5. It is found that a significant reduction in diffraction coefficient may occur as the pit dimensions increase. In addition, back-scattering from the pit results in a partial standing wave system in front of the pit, this leads to the maximum diffraction coefficient becoming larger and the minimum diffraction coefficient being reduced greatly in the shadow region. As seen from Fig. 21 and Fig. 22 that, for pits with a dimensionless water depth \( d/h \) is 2.5 and 2 respectively and dimensionless width \( w_2/L \) greater than 2.5; the minimum diffraction coefficient and \( w_2/L \) greater than 0.5 the maximum diffraction coefficient approaches a steady value.

Fig. 18—Variation of maximum and minimum diffraction coefficient with dimensionless pit length for \( w_1/L = 1 \), \( d/h = 2.5 \), \( kh = 0.167 \). Notations: minimum diffraction coefficient (ooo), maximum diffraction coefficient (***).

Fig. 19—Variation of maximum and minimum diffraction coefficient with dimensionless pit length for \( w_1/L = 0.5 \), \( d/h = 3 \), \( kh = 0.167 \). Notations: minimum diffraction coefficient (ooo), maximum diffraction coefficient (***).

Fig. 20—Variation of maximum and minimum diffraction coefficient with dimensionless pit length for \( w_1/L = 0.374 \), \( d/h = 3 \), \( kh = 0.167 \). Notations: minimum diffraction coefficient (ooo), maximum diffraction coefficient (***).

Fig. 21—Variation of maximum and minimum diffraction coefficient with dimensionless pit length for \( w_1/L = 0.5 \), \( d/h = 2.5 \), \( kh = 0.167 \). Notations: minimum diffraction coefficient (ooo), maximum diffraction coefficient (***).
Conclusion

A nonlinear dispersive wave model with inclusion of a semi-empirical wave breaking effect, valid for arbitrary water depths is given in the boundary fitted curvilinear coordinate system. To analyze the diffraction of surface waves with submarine pits in water of nonuniform depth and the numerical code of these equations is developed. Comparisons for the test cases support the reliability and accuracy of the wave model and the numerical approach. Simulation of waves over an asymmetrical trapezoidal bar, over an elliptic shoal, and over a circular shoal is compared with the experiments and showed a good level of agreement for nonlinear waves. Various numerical simulations have been presented to show the influence of numerous submarine pit parameters on the wave field for several example configurations. Due to backscattering from the submarine pit; in lateral and front sides of the pit a partial standing wave system develops, while in the lee of the pit a reduced wave height shadow zone exists, the maximum diffraction coefficient becomes larger and the minimum diffraction coefficient is significantly reduced. While the value of d/h increases, the level of protection provided by the pit also increases.

Results of the numerical model have been verified through comparisons with those obtained by previous works and agreement is observed. Use of contravariant velocities provides important numerical advantages; the transformed equations are not too complicated therefore the numerical treatment is simple. Although the examples presented here do not require the use of boundary fitted coordinates, this work aims to give a more general formulation which can be applied. It is concluded that the present numerical wave model accurately simulates the wave field due to wave interaction with submarine pits in water of nonuniform depth and so the model gives confidence for practical coastal engineering applications. Overall, the wave model with breaking dissipation terms can simulate nonlinear breaking waves over arbitrary bathymetries and lateral boundaries surrounding the domain. In the future, the submarine pits with slopes will be investigated. Method can be extended to include a surface roller type approximation for breaking of waves.

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References


