Buckling of an orthotropic cylindrical thin shell with continuously varying thickness under a dynamic loading

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The buckling of an orthotropic composite cylindrical shell with variable thickness, subjected to a dynamic loading, is reported here. At first, the fundamental relations and Donnell type dynamic buckling equation of an orthotropic cylindrical shell with variable thickness have been obtained. Then, employing Galerkin’s method, these equations have been reduced to a time dependent differential equation with variable coefficients. Finally, for different initial conditions and approximation functions, applying the Ritz type variational method, analytical expression has been found for the dynamic factor. Using these results, the effect of the variations of the power of time in the external pressure expression, the loading parameter and the ratios of the Young’s moduli on the dynamic factor are studied numerically for the case when the thickness of the cylindrical shell varies as a power and exponential functions. It has been observed that these effects change the dynamic factor of the problem in the heading appreciably.

Shells are one of the necessary structural components and commonly found in a variety of engineering applications, such as the military, aerospace, turbo machinery and ship building industries. In recent years, new composite materials made of two or more components are used in many fields of industry. Shells composed of composite materials constitute different structures having orthotropic characteristics.

There are numerous methods to determine the static critical loads, consistent with experimental results for uniform orthotropic composite thin shells under different loads with different boundary conditions. There are some reports about orthotropic composite thin shells with variable thickness due to the difficulties in their production and theoretical analysis. None-the-less, it is highly probable that this type of structural parts will be used a lot in the future due to the advantages of their light weights and small volume and the progress in their fabrication methods. In recent years, numerous research works have been reported the buckling and vibration of shells with variable thickness.

The effect of the variation of thickness on the dynamic factor has not been studied appreciably. The solution of a dynamic problem is reduced to the determination of the dynamic factor for certain loading cases. The dynamic factor can be found, using different methods, depending on the manner in which the loading is applied, particularly on the loading parameter.

The aim of the present study is to investigate the buckling of an orthotropic composite cylindrical shell with continuously varying thickness subjected to a uniform external pressure which is a power function of time, for different initial conditions and approximation function using the Ritz type variational method.

Theoretical

Equations of motion

Consider a circular cylindrical thin shell of medium length composed of orthotropic composite materials, with immovable simple supports at the ends, having length $L$, radius $r$ and thickness $h$. The right-handed system of coordinates is selected in such a way that the origin is on the middle surface, the $Ox_3$ axis is perpendicular to the middle surface of the plane, positive inwards, and the $Ox_1$ and $Ox_2$ axes are in the axial and circumferential directions, respectively (Fig. 1). $h$ is a continuous and second order differentiable function with respect to in the axial direction.
Taking account of the radial inertia forces, the modified Donnell type basic equations of cylindrical shell are found as:

\[ \begin{align*}
\frac{\partial}{\partial x_1} \left( n_{11} \right) + 2 n_{12} \frac{\partial}{\partial x_2} + n_{22} \frac{\partial}{\partial x_1} + n_{11} \frac{\partial}{\partial x_2} + & \frac{n_{22}}{r} = \rho u_{11} \quad \ldots (1) \\
\frac{\partial^2}{\partial x_1^2} e_{11}^0 + \frac{\partial^2}{\partial x_2^2} e_{22}^0 - 2 \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} e_{12}^0 = -u_{11} \frac{\partial}{\partial x_1} \quad \ldots (2)
\end{align*} \]

where \( n_{11} , \ n_{22} \) and \( n_{12} \) are axial, circumferential and twisting moments, respectively, \( n_{11}^0 , n_{22}^0 \) and \( n_{12}^0 \) are incremental stress resultants about fundamental state, \( u \) is the displacement of the middle surface of the shell in the normal direction, positive inwards, \( e_{11}^0 \) and \( e_{22}^0 \) are the normal strains in the directions of the axial and circumferential axes, respectively, and \( e_{12}^0 \) is the shear strain (all on the same surface), (.) denoting differentiation with respect to, \( \rho \) is material density and \( t \) the time.

The well-known force and moment resultant are expressed by:

\[ [n_{11} , n_{22} , n_{12}] , [m_{11} , m_{22} , m_{12}] \]

\[ = \int_{-h/2}^{h/2} \left[ e_{11}^0 , e_{22}^0 , e_{12}^0 \right] dx_3 \quad \ldots (3) \]

The relation between the forces and the Airy stress function \( \varphi = \varphi / h_0 \) is given by:

\[ [n_{11} , n_{22} , n_{12}] = \left[ \varphi_{x_2} , \varphi_{x_3} , -\varphi_{x_1} \right] \quad \ldots (4) \]

where \( h_0 \) is the nominal thickness of the shell.

But, the Hooke's Law for an orthotropic material is:

\[ \sigma_{11} = \frac{E_{11} L_1}{h_0} \left( \begin{array}{c} Q_{11} \\ Q_{12} \\ Q_{13} \end{array} \right) , \quad \sigma_{22} = \frac{E_{22} L_2}{h_0} \left( \begin{array}{c} Q_{21} \\ Q_{22} \\ Q_{23} \end{array} \right) , \quad \sigma_{12} = \frac{G_{12} L_3}{h_0} \left( \begin{array}{c} Q_{11} \\ Q_{21} \end{array} \right) \quad \ldots (5) \]

Therefore,

\[ Q_{11} = \frac{E_{11}}{1 - \mu_{12} \mu_{21}} , \quad Q_{22} = \frac{E_{22}}{1 - \mu_{12} \mu_{21}} , \quad Q_{12} = \mu_{21} Q_{11} = \mu_{12} Q_{22} \quad \ldots (6) \]

in which \( E_{11} \) and \( E_{22} \) are the Young's moduli in the directions of the \( Ox_1 \) and \( Ox_2 \) axes, \( \mu_{12} \), \( \mu_{21} \) are the Poisson's ratios of the orthotropic material and \( \mu_{21} E_{11} = \mu_{12} E_{22} \).

The shell is subjected to an external pressure varying as a power function of time:

\[ n_{11}^0 = 0 , \quad n_{22}^0 = -ho h t^q , \quad n_{12}^0 = 0 \quad \ldots (7) \]

where \( \rho \) is the loading parameter and \( q \) is the power of time in the external pressure expression, the last one being equal to or greater than unity.

After substituting Eqs (3-7) into relations (1) and (2), and using dimensionless coordinates \( \xi_1 = x_1 / L \), \( \xi_2 = x_2 / r \), we get the system of matrix equations:

\[ \left[ L_{11} \ L_{12} \right] \left[ u \right] = 0 \quad \ldots (8) \]

where the differential operators \( L_{mn} (m, n = 1,2) \) appearing in Eq. (8) are defined according to:

\[ L_{11} = \frac{\partial^4}{\partial \xi_1^4} + \frac{\mu_{21} L_1^2}{\mu_{12} r^2} \frac{\partial^4}{\partial \xi_1^2 \partial \xi_2^2} + \frac{\mu_{12} L_1^4}{\mu_{21} r^4} \frac{\partial^4}{\partial \xi_2^4} + \frac{2}{h^2} \frac{d}{d \xi_1} \left( \frac{\partial^3}{\partial \xi_1^3} + \frac{\mu_{21} L_1^2}{\mu_{12} r^2} \frac{\partial^3}{\partial \xi_1 \partial \xi_2^2} \right) + \frac{1}{h^2} \frac{d}{d \xi_2} \left( \frac{\partial^2}{\partial \xi_1^2} + \frac{\mu_{12} L_1^2}{\mu_{21} r^2} \frac{\partial^2}{\partial \xi_2^2} \right) \times \left[ \frac{\rho h}{r} \frac{\partial^2}{\partial \xi_2^2} + \rho \frac{\partial^2}{\partial t^2} \right] \quad \ldots (9) \]

\[ L_{12} = \frac{12(1 - \mu_{12} \mu_{21}) L_1^2}{E_{11}} \frac{\partial^2}{\partial \xi_1^2} \frac{\partial^2}{\partial t^2} \quad \ldots (10) \]
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Solution of eigenvalue problem

Assuming the shell to be simply supported along the peripheries of both bases, the solution of the system of Eq. (8) is sought as follows:  

\[
\phi = \sum_{i,j} \phi_{ij}(t) \sin j\pi \xi_1 \sin i\xi_2,
\]

\[
u = \sum_{i,j} \nu_{ij}(t) \sin j\pi \xi_1 \sin i\xi_2
\]

where \( j \) is the number half-waves along the shell length at buckling in the axial direction, \( i \) denotes the number of waves circumferential direction during buckling and \( \phi_{ij}(t) \) and \( \nu_{ij}(t) \) are time dependent amplitudes. Substituting expressions (13) in Eq. (8) and then applying Galerkin's method in the ranges \( 0 \leq \xi_1 \leq 1 \) and \( 0 \leq \xi_2 \leq 2\pi \) in obtained equation set, the terms that are included by expression (13) are solved. One of these terms is selected and eliminating \( \phi_{ij}(t) \) from the equations, thus obtained, and remembering that when the half wave number \( j \) is equal to one, the wave number \( i \) for a shell of medium length satisfies the inequality \( i^4 \gg \delta^4 \), the following equation is obtained \(^{11,12}\):

\[
d^2u_{ij}(\tau) + \tau^2 u_{ij}(\tau) = 0
\]

where \( \delta = \pi r / L \), \( \tau = \tau_{cr} \), in which \( \tau_{cr} \) is the critical time and the dimensionless time parameter \( \tau \) satisfies \( 0 \leq \tau \leq 1 \), and the following definitions apply:

\[
\theta_1 = \int_0^1 h^3(\xi_1) \sin^2 \pi \xi_1 d\xi_1, \quad \theta_2 = \int_0^1 h(\xi_1) \sin^2 \pi \xi_1 d\xi_1,
\]

\[
\rho_1 = \rho \int_0^1 h(\xi_1) \sin^2 \pi \xi_1 d\xi_1
\]

Eq. (14) is solved for two initial conditions. (i) In first approximation, the function satisfying initial conditions \( u_0(0) = 0, u_{0,\tau}(0) = 0 \) is in the form:

\[
u_{ij}(\tau) = A_{ij} u_0(\tau) = A_{ij} e^{\delta \tau^2} [53/52 - \tau] \quad (16)
\]

(ii) Furthermore, the curve \( (u_0, \tau) \) has the maximum when \( \tau = 1 \), so in first approximation the function satisfying initial conditions \( u_0(0) = 0, u_{0,\tau}(1) = 0 \) is in the form:

\[
u_{ij}(\tau) = A_{ij} u_0(\tau) = A_{ij} e^{\delta \tau^2} [52/51 - \tau] \quad (17)
\]

where \( A_{ij} \) is amplitude \(^{11,12}\).

Applying the Ritz type variational method to differential Eq. (14) and minimizing characteristic equation according to the wave number \( i \), an equation is obtained. After solving this equation and after mathematical operations, the following expression for dynamic factor is found \(^{11,12}\):

\[
K_d = 1.1398 \times \left[ \frac{4\rho_1 \rho_0^{2/3} \xi(1+3\xi)^{3/4} \theta_1^{5/4} \theta_2^{3/4}}{E_{22} \theta_1} \right]^{2/3} \left[ \frac{3(1-\mu_{12}\mu_{21})}{E_{22} \theta_1} \right]^{1/3} \left( \frac{\lambda_0}{\lambda_1} \right)^{1/3} \left( \frac{\lambda_2}{\lambda_1} \right)^{1/3} \right]^{1/4}
\]

where \( \lambda_0, \lambda_1 \) and \( \lambda_2 \) are defined as:

\[
\lambda_0 = \int_0^1 [u_{ij}(\tau)]^2 d\tau, \quad \lambda_1 = \int_0^1 [u_{0,\tau}(\tau)]^2 d\tau,
\]

\[
\lambda_2 = \int_0^1 [\tau^2 u_{ij}(\tau)]^2 d\tau
\]

When \( h = \) constant from the expressions (18) for a uniform orthotropic composite cylindrical shell, the appropriate formula is found as a special case.

Results and Discussion

The case of an orthotropic composite cylindrical thin shell in which the thickness changes as a power
and exponential functions of time has been considered by Irie et al. The thickness at the two ends being \( h_1 \) and \( h_2 \), this case can be expressed in the following manner:

\[
h = h_2 - (h_2 - h_1)(1 - \xi^k) \quad k > 0 \quad \ldots (20)
\]

\[
h = h_1 (h_2 / h_1)^{\xi^k} \quad \ldots (21)
\]

where \( k = 1, 2, 3 \ldots \) correspond to the cases of linear, parabolic, cubic etc. variations, respectively (Fig. 1).

Numerical computations have been carried out for the following data used in the computations, Graphite/epoxy material properties\(^{7,18}\):

- \( E_{11} = 1.724 \times 10^5 \) MPa
- \( E_{22} = 7.79 \times 10^3 \) MPa
- \( \mu_{12} = 0.35 \)
- \( h_2 = 8 \times 10^{-4} \) m
- \( r = 9 \times 10^{-2} \) m
- \( L = 0.2 \) m

Using above data, numerical results found for various loading parameters \( p \) and power of time \( q \) values have been presented in graphical and tabular forms. The approximation function (17) is used in

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Table 1. The difference between dynamic factors must be 1% in approximation functions. Consequently, for two of the approximation functions nearly the same solutions are obtained. For uniform or non-uniform thickness, when \( p \) increases, the value of dynamic factor increases. But when \( q \) increases, the value of dynamic factor decreases. For small variations of thickness, it is observed that the difference between dynamic factors of the shells that have uniform and non-uniform thickness is important as shown in Table 1. For example, when \( q=1, p=100 \text{ MPa/s} \) and \( h_1/h_2=0.75 \), the effect of linear variation of thickness to dynamic factor is 17.5%.

For validity of the analysis, comparisons with isotropic and orthotropic cylindrical shells with uniform thickness under external pressure varying linearly dependent on time, are made. Comparisons were carried out for the following isotropic material, shell and loading properties: \( E=77500 \text{ MPa}, \mu =0.3 \), \( \rho =310 \text{ kgx}^2\text{m}^4/\text{m}^6, p=650 \text{ MPa/s}, q =1 \) and orthotropic material, shell and loading properties: \( E_{11}/E_{22} =2, \mu_{12}=0.12, \rho =184 \text{ kgx}^2\text{m}^4, h/R=2/143, L/R=2.6, q=1, p=250 \text{ MPa/s}. \)

The present theory gives dynamic factor for isotropic material \( K_d=4.248 \). Experimental value for the same given by Agamirov,\(^{13}\) is \( K_d=4.0 \) and numerical result given by Shumikh,\(^{14}\) is \( K_d=4.239 \). The present theory gives dynamic factor for orthotropic material \( K_d=4.32 \) and numerical result given by Ogibalov et al.,\(^{15}\) is \( K_d=4.0 \). It is observed that the results are harmonic.

Conclusions

The buckling of an orthotropic composite thin shell with continuously varying thickness, subjected to a dynamic loading, has been studied. At first, the fundamental relations and modified Donnell type dynamic buckling equations have been written for an orthotropic composite shell with continuously varying thickness subject to an external pressure which is a power function of time. Then, applying Galerkin's method, a time dependent differential equation with variable coefficient has been obtained. Finally, for different initial conditions and approximation functions, applying the Ritz type variational method, analytical expression has been found for the dynamic factor. Using these results, the effects of the variations of the power of time in the external pressure expression, the ratios of the Young's moduli and the loading parameter on the dynamic factor are studied, numerically, for the case when the thickness of the cylindrical shell varies as a power and exponential function. Furthermore, the present method has been verified by comparisons of dynamic factor with the theoretical and experimental ones given in previous literature for the case of a shell with uniform thickness subject to a uniform external pressure, which is a linear function of time.

**Nomenclature**

\[
\begin{align*}
E_{11}, E_{22} &= \text{Young's moduli of the orthotropic material} \\
E &= \text{Young's modulus of the isotropic material} \\
\epsilon_0, \epsilon_0, \epsilon_2 &= \text{Axial, circumferential and shear strains on the middle surface of the shell} \\
h, h_0 &= \text{Thickness of the shell and nominal thickness of the shell, respectively} \\
h_1, h_2 &= \text{Thickness of the shell at the two ends} \\
i, j &= \text{Wave numbers in the circumferential and axial directions, respectively} \\
k &= \text{Power coefficient of thickness variation} \\
K_d &= \text{Dynamic factor} \\
L &= \text{Length of the cylindrical shell} \\
L_{nn} &= \text{Differential operators defined in Eqs (9-12)} \\
n_{11}, n_{22}, n_{12} &= \text{Axial, circumferential and twisting moments, respectively} \\
n_{11,0} n_{22,0} n_{12,0} &= \text{Incremental stress resultants about fundamental state} \\
p &= \text{Loading parameter} \\
q &= \text{Power of time in the external pressure expression} \\
r &= \text{Radius of the cylindrical shell} \\
t, t_r &= \text{Time and critical time, respectively} \\
u &= \text{Displacement of the middle surface in the inwards normal direction} \\
v_0(t) &= \text{Time dependent amplitude} \\
\delta &= \pi r/L \\
\phi, \phi_0(t) &= \text{Stress function and time dependent amplitude, respectively} \\
\lambda_0, \lambda_1, \lambda_2 &= \text{Coefficient defined in Eq. (19)} \\
\mu &= \text{Poisson's ratio of the isotropic material} \\
\mu_{12}, \mu_{21} &= \text{Poisson's ratios of the orthotropic material} \\
\theta_1, \theta_2, \theta_3 &= \text{Coefficient defined in Eq. (15)} \\
\sigma_{11}, \sigma_{12}, \sigma_{22} &= \text{Stress components} \\
\tau &= \text{Dimensionless time parameter} \\
\rho &= \text{Density of the material} \\
x_1/L &= \text{Dimensionless axial coordinate} \\
x_2/r &= \text{Dimensionless circumferential coordinate} \\
(,) &= \text{Denoting differentiation with respect to}
\end{align*}
\]
References