Computation of stress intensity factors using a special finite element technique

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The use of a special finite element technique for the computation of stress intensity factors in cracked plates, subjected to in-plane tensile loads, is described here. Seven plates of unit thickness (each 40 x 40 cm²) are numerically analyzed. Each of the plates contains an edge-crack of varying length (4-28 cm). Firstly, the plates are conceptually divided into two zones, a circular singular zone around the crack-tip and a conventional zone outside the circular boundary. The plates are then discretized into a system of triangular finite elements. The elements near the crack-tip are made much smaller than the remote ones. Appropriate displacement functions are proposed for handling the inverse square root singularity of stresses at the crack-tip. Also, the conventional elements are used outside the singular zone. Based on the proposed displacement functions, a new stiffness matrix is derived. A suitable computer program is developed for the analysis of stresses in the cracked plates. The stress intensity factors are computed for each of the seven plates and plotted against the ratio of crack-length to plate-length. The results obtained by using the proposed displacement functions are found to be more satisfactory than those obtained by the conventional elements using linear displacement functions.

In the theory of elasticity, the stress behaviour in the neighbourhood of a crack-tip is of special importance. Many investigators have studied the elastic-stress distributions around the crack-tips. Some of the earliest such contributions are by Griffith¹, who set up an energy criterion for crack instability, and Williams², who showed that the stresses vary inversely as the square root of the distance from the tip at the tip of a crack. Irwin³ obtained a series of linear elastic stress field solutions for a plane crack and showed that the stress intensity factor parameter is proportional to the square root of the force tending to cause crack extension.

Many researchers, viz. Walsh⁴, Yamamoto and Tokuda⁵, Benzley⁶, Atluri et al.⁷, Gifford and Hilton⁸, Heyniger and Kriz⁹, Shivakumar and Raju¹⁰, etc. have presented various special finite element techniques for the computation of stress intensity factors at a crack-tip. The present studies involve the introduction of a new set of displacement functions for the finite element analysis of stresses in a cracked plate. The proposed displacement functions have been found to be quite adequate to represent the inverse square root singularity of stresses near the crack-tip.

**Computational Strategy**

The cracked plate is conceptually divided into two zones: the circular singular zone having its centre at the crack-tip; and the surrounding regular zone outside the circular boundary. The plate is then discretized into a system of triangular finite elements. The mesh in the vicinity of a crack-tip is made fine enough to allow adequate representation of the stress concentration in the neighbourhood of the tip (Fig. 1). The mesh away from the crack-tip is composed of relatively larger elements. Such a differential treatment requires less computer memory as well as

![Fig. 1—Finite element subdivision of a cracked plate with R as radius of the inner circle](image-url)
less execution time. The elements lying within the circular singular zone are treated as special elements whereas those outside the circular boundary are treated as conventional ones. The compatibility of the displacement functions between the singular zone and the conventional zone is maintained. The computer program, developed on the basis of proposed displacement functions, is executed on the CDC 6000 computer. The fundamental sets of data required in the internal operations of the finite element program are:

a. Junction coordinates—The coordinates of all the nodal points are listed in sequential order.
b. Element characteristics—The nodal numbers associated with each element, and a type number representing the material properties of that element are listed sequentially.
c. Boundary conditions—The nodal boundary conditions are read in as combinations of 0s and 1s for each degree of freedom with '0' denoting a free degree of freedom and '1' denoting a fixed degree of freedom.
d. Loads—The loads may be read in as a single vector. When the loaded nodes form only a small percentage of total nodes in the system, it is advantageous to read in only the non-zero loads together with their nodal numbers. The load vector would of course be set to zero beforehand to take care of the unloaded nodes.
e. Radius of the circular singular zone around the crack-tip.

The program includes a stiffness generation routine which is solely dependent upon the basic mathematical formulation for element stiffness. In particular, it depends upon whether numerical integration is needed for the element (the special elements located in the vicinity of the crack-tip) or whether direct integration is possible for the conventional elements lying outside the singular zone. The stiffness generation routine is also used to generate the stress matrix, which produces the element stresses when multiplied by the relevant nodal displacements. This matrix is a by-product of the stiffness generation and a little computer time is required to generate it. As regards the assembly and solution of equations, a routine based on the Gauss elimination procedure has been used.

The program computes the displacements at the nodes and stresses at the centroid of each element. The stress intensity factor for each cracked plate is then computed from the known values of stress at the centroid of the tip element which lies along the axis of the crack. Also, the stress fields at the tip of a crack, described by the well-known equations developed by Irwin, are:

\[
\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \left[ \cos \left( \frac{\theta}{2} \right) \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \right]
\]

\[
\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \left[ \cos \left( \frac{\theta}{2} \right) \left( 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \right]
\]

\[
\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \left[ \cos \left( \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \right]
\]

where \( \sigma_{xx} \) is the normal stress in the x-direction, \( \sigma_{yy} \) is the normal stress in the y-direction, \( \tau_{xy} \) is the shearing stress in the xy-plane, \( K_I \) is the stress intensity factor for Mode I crack extension, \( r \) is the distance of a point from the crack-tip and \( \theta \) is the angle made by a point with respect to the crack-plane.

As \( \theta = 0 \) for the tip element which lies along the axis of the crack, Irwin's equations reduce to the simple form:

\[
\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}
\]

\[
\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}
\]

\[
\tau_{xy} = 0
\]

Hence, the stress intensity factor \( (K_I) \) can be easily computed.

The Finite Element Model

The displacement components within a conventional triangular element can be simply represented by the linear polynomials as described by Eq. (1):

\[
u = A_1 + A_2x + A_3y
\]

\[
u = A_4 + A_5x + A_6y
\]
The six constants, $A_1, A_2, A_3, A_4, A_5$ and $A_6$, can be evaluated easily by solving the two sets of three simultaneous equations which arise if the nodal coordinates are inserted and the displacements equated to the appropriate nodal displacements. Now, let the displacements within a special element be represented by the set of polynomials given by Eq. (2):

$$
u = \sqrt{\frac{R}{r}} \left( A_3 x + A_5 y \right) + A_4$$

where $R$ is the radius of the circular singular zone around the crack-tip.

In polar coordinates,

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Therefore,

$$x^2 + y^2 = r^2$$

$$\left( x^2 + y^2 \right)^{\frac{1}{4}} = r^{\frac{1}{2}}$$

Combining Eqs (2) and (3), we get Eqs (4):

$$u = A_1 + \sqrt{\frac{R}{r}} \left( A_2 x + A_4 y \right) \left( x^2 + y^2 \right)^{\frac{1}{4}}$$

$$v = A_4 + \sqrt{\frac{R}{r}} \left( A_3 x + A_5 y \right) \left( x^2 + y^2 \right)^{\frac{1}{4}}$$

Differentiating $u$ and $v$ with respect to $x$ and $y$ we get:

$$\frac{\partial u}{\partial x} = A_2 \left( g + h x^2 \right) + A_3 h x y$$

$$\frac{\partial u}{\partial y} = A_4 h x y + A_5 \left( g + h y^2 \right)$$

$$\frac{\partial v}{\partial x} = A_3 \left( g + h x^2 \right) + A_5 h x y$$

$$\frac{\partial v}{\partial y} = A_4 h x y + A_6 \left( g + h y^2 \right)$$

where

$$g = \sqrt{R \left( x^2 + y^2 \right)^{\frac{1}{4}}}$$

$$h = -\frac{\sqrt{R}}{2} \left( x^2 + y^2 \right)^{\frac{3}{4}}$$

The strain components are given by Eqs (7):

$$e_{xx} = \frac{\partial u}{\partial x}$$

$$e_{yy} = \frac{\partial v}{\partial y}$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

From Eqs (5) and (7), we get Eqs (8):

$$e_{xx} = A_2 \left( g + h x^2 \right) + A_3 h x y$$

$$e_{yy} = A_4 h x y + A_5 \left( g + h y^2 \right)$$

$$e_{xy} = A_3 \left( g + h x^2 \right) + A_5 h x y + A_6 \left( g + h y^2 \right) + A_6 h x y$$

Eqs (8) can be written in matrix form as Eq. (9):

$$\begin{align*}
\{ \varepsilon \} &= [N] [A] \\
&= \begin{bmatrix} 0 & (g + h x^2) & h x y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h x y & (g + h y^2) \\
0 & h x y & (g + h y^2) & 0 & (g + h x^2) & h x y \end{bmatrix} [A] \\
&= \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{bmatrix}^T
\end{align*}$$

Let $a, b$ and $c$ be the nodes of the $i$th element. Then the nodal displacements are:

$$u_a = A_1 + \sqrt{\frac{R}{r}} \left( A_2 x_a + A_3 y_a \right) \left( x_a^2 + y_a^2 \right)^{\frac{1}{4}}$$

$$v_a = A_4 + \sqrt{\frac{R}{r}} \left( A_3 x_a + A_5 y_a \right) \left( x_a^2 + y_a^2 \right)^{\frac{1}{4}}$$

$$u_b = A_1 + \sqrt{\frac{R}{r}} \left( A_2 x_b + A_3 y_b \right) \left( x_b^2 + y_b^2 \right)^{\frac{1}{4}}$$

$$v_b = A_4 + \sqrt{\frac{R}{r}} \left( A_3 x_b + A_5 y_b \right) \left( x_b^2 + y_b^2 \right)^{\frac{1}{4}}$$

$$u_c = A_1 + \sqrt{\frac{R}{r}} \left( A_2 x_c + A_3 y_c \right) \left( x_c^2 + y_c^2 \right)^{\frac{1}{4}}$$

$$v_c = A_4 + \sqrt{\frac{R}{r}} \left( A_3 x_c + A_5 y_c \right) \left( x_c^2 + y_c^2 \right)^{\frac{1}{4}}$$
Eqs (13) can be expressed in matrix form as:

\[
\{\delta\} = [C]\{A\} 
\]  \hspace{1cm} \ldots (14)

where

\[
\{\delta\} = \begin{bmatrix} u_{xi} & v_{xi} & u_{yi} & v_{yi} & u_{zi} & v_{zi} \end{bmatrix}^T
\]

\[
[C] = \begin{bmatrix} 1 & px_a & py_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & px_a & py_a \\ 1 & qx_b & qy_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & qx_b & qy_b \\ 1 & sx_e & sy_e & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & sx_e & sy_e \end{bmatrix}
\]  \hspace{1cm} \ldots (15)

\[p = \sqrt{R(x_a^2 + y_a^2)} \] \hspace{1cm} \ldots (16)

\[q = \sqrt{R(x_b^2 + y_b^2)} \] \hspace{1cm} \ldots (17)

\[s = \sqrt{R(x_e^2 + y_e^2)} \] \hspace{1cm} \ldots (18)

Multiplying both sides of Eq. (14) by \([C]^{-1}\) we get

\[\{A\} = [C]^{-1} \{\delta\} \] \hspace{1cm} \ldots (18)

Combining Eqs (9) and (18), we get Eq. (19):

\[\{\varepsilon\} = [N][C]^{-1} \{\delta\} \] \hspace{1cm} \ldots (19)

The well-known relationship described by Zienkiewicz \cite{12} is given by Eq. (20):

\[\{\varepsilon\} = [B] \{\delta\} \] \hspace{1cm} \ldots (20)

Comparing Eqs (19) and (20), we get Eq. (21):

\[[B] = [N][C]^{-1} \] \hspace{1cm} \ldots (21)

Let \([C]^{-1} = [X]\) \hspace{1cm} \ldots (22)

Therefore,

\[[B] = [N][X] \] \hspace{1cm} \ldots (23)

The standard expression for the stiffness matrix is given by Eq. (24):

\[ [K_1] = \int \Delta [B]^T [D][B] \cdot t \, dxdy \] \hspace{1cm} \ldots (24)

where \(t\) is the thickness of the element and the integration is taken over the area of the triangle \(\Delta\).

Combining Eqs (23) and (24), we finally get Eq. (25):

\[ [K_1] = \int \Delta [X]^T [N]^T [D][N][X] \cdot t \, dxdy \] \hspace{1cm} \ldots (25)

Results and Discussion

Computational experiments are performed on seven plates, each of dimension 40 x 40 cm\(^2\) and of unit thickness. Each plate has an edge-crack whose length is varied from 4 to 28 cm. The cracked plate, idealized with 216 triangular elements and 134 nodes, is loaded with 2.75 MPa under tension. Fig. 2 shows a typical plate with an edge-crack which lies along the central line of the plate. A uniformly distributed tensile load is applied in a direction normal to the crack plane. So, all the computational experiments reported here, have been carried out for Mode I crack extension.

The values of the stress intensity factors, computed from the conventional elements as well as the special
elements, are plotted against the ratio of crack-length to plate-length, $R'$ (Fig. 3). The values of the stress intensity factors given by Rooke and Cartwright\textsuperscript{13} have been found to differ from those of the conventional elements (Fig. 4). This variation is more or less the same as the variation between the results of the special elements and those of the conventional elements shown in Fig. 3. No significant difference between the reported results\textsuperscript{13} and those from the special elements is shown by Fig. 5. Thus, the values of the stress intensity factors obtained from the proposed displacement functions agree closely with the reported results\textsuperscript{13}. Fig. 6 compares the values of the stress intensity factors obtained from Broek\textsuperscript{14}, and Rooke and Cartwright\textsuperscript{13} for the special elements and the conventional elements. A satisfactory agreement

Fig. 3—Comparison of stress intensity factors obtained from conventional and special elements for an edge-crack

Fig. 5—Comparison of stress intensity factors obtained from special elements and those from Rooke and Cartwright for an edge-crack

Fig. 4—Comparison of stress intensity factors obtained from conventional elements and those from Rooke and Cartwright for an edge-crack

Fig. 6—Comparison of stress intensity factors obtained from conventional and special elements and those from Rooke and Cartwright and from Broek for an edge-crack
is observed among the first three results. Figs 3 to 6, therefore, establish that the proposed displacement functions represent the stress singularity at the crack tip much better than the linear functions of the conventional elements.

The present computational experiments need only a small storage capacity of the computer as the number of elements for adequate representation of the discretized plates is only 216. This economic use of computer memory may be attributed to the simplicity of the displacement functions and hence the stiffness matrix.

Conclusions

The present investigations describe the use of a new set of displacement functions for the triangular finite elements in the vicinity of a crack-tip. The proposed displacement functions adequately represent the inverse square root singularity of stresses near the crack-tip. The compatibility of the displacement functions between the singular zone and the conventional zone is maintained. A new stiffness matrix is derived and a suitable computer program is developed on the basis of these displacement functions. Computational experiments are performed with edge-cracked plates for establishing the credibility of the present scheme of solution of crack problems. The special finite element analysis with the proposed displacement functions provides satisfactory results for the stress intensity factors.

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Nomenclature

\( A_1, A_4, \ldots, A_k \) = unknown constants of the displacement functions
\( a, b, c \) = nodes of the triangular element
\( \{b\} \) = strain distribution function
\( \{D\} \) = elasticity matrix
\( d \) = crack-length
\( \epsilon_{xx} \) = normal strain in the x-direction
\( \epsilon_{yy} \) = normal strain in the y-direction
\( \epsilon_{xy} \) = shear strain in the xy-plane
\( i \) = element number
\( K_i \) = stress intensity factor for Mode I crack extension
\( [K] \) = element stiffness matrix
\( L \) = plate-length
\( R \) = radius of the circular singular zone around the crack-tip
\( R' \) = ratio of the crack-length to plate-length
\( r \) = distance of a point from the crack-tip
\( t \) = plate thickness
\( u \) = displacement component in the x-direction
\( v \) = displacement component in the y-direction
\( \Delta \) = area of the triangle
\( \{d\} \) = nodal displacement vector
\( \{E\} \) = elastic strain vector
\( \theta \) = angle made by a point with respect to the crack-plane
\( \sigma_{xx} \) = normal stress in the x-direction
\( \sigma_{yy} \) = normal stress in the y-direction
\( \tau_{xy} \) = shearing stress in the xy-plane

References

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