Non-linear propagation of electrostatic waves in relativistic Fermi plasma with arbitrary temperature

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Modulational instability of electron plasma waves has been investigated by using the quantum hydrodynamic (QHD) model for quantum plasma at finite temperature by deriving a non-linear Schrödinger equation including relativistic effects. It is shown that the electron degeneracy parameter and streaming velocity significantly affect the linear and non-linear properties of electron plasma waves in a finite temperature quantum plasma.

Keywords: Electron plasma waves, Modulational instability, Quantum plasma, QHD model, Relativistic effect

1 Introduction
Many researchers have studied the linear and non-linear properties of waves in classical non-relativistic plasma. However, particle velocities in some plasmas may become comparatively high; in some cases it may even approach the speed of light. For such plasmas, it becomes important to take into account the relativistic effects. In fact relativistic effect may significantly modify the linear and non-linear behaviour of plasma waves. Relativistic plasma can be formed in many practical situations e.g. in space-plasma phenomena¹, the plasma sheet boundary of earth’s magnetosphere², Van Allen radiation belts³ and laser-plasma interaction experiments⁴. The relativistic motion in plasmas is assumed to exist during the early period of evolution of the universe⁵. Regarding the relativistic effects on ion-acoustic solitary waves, a number of research works have been reported. Das and Paul⁶ first investigated the ion-acoustic solitary waves in relativistic plasma and showed that relativistic effect is important on the formation of ion-acoustic solitary waves in the presence of streaming motion of plasma particles. Subsequently, many researchers considered various parameters together with the relativistic effect to study the ion-acoustic solitary waves and obtained some fascinating results which are important in laboratory and space plasma. Recently, Saeed et al⁷, have shown that in electron-positron-ion plasma, increase in the relativistic streaming factor causes the soliton amplitude to thrive and its width to shrink. El-Labany et al⁸, have shown that relativistic effect can modify the condition of modulational instability of ion-acoustic waves in warm plasma with non-thermal electrons. Han et al⁹, have studied the existence of ion-acoustic solitary waves and their interaction in weakly relativistic two-dimensional thermal plasma. Electron acoustic solitons in relativistic plasma with non-thermal electrons have been studied by Sahu and Roy Chowdhury¹⁰. Gill et al¹¹, have studied the amplitude and width variations of ion-acoustic solitons in relativistic electron-positron-ion plasma. Recently, Bharuthram and Yu¹² have shown that relativistic electron plasma waves can propagate as quasi-stationary non-linear waves as well as solitary waves. All of the above research works on the relativistic effects on plasma waves have been reported for classical plasma. But in plasmas, where the particle density is high and the temperature is low quantum phenomenon becomes important. Such quantum plasma is ubiquitous in white dwarfs, neutron stars, galactic plasma, metal nanostructures, intense laser-solid interaction and in many other environment. In recent years, propagation of various electrostatic modes such as ion-acoustic waves, electron-acoustic waves, dust-acoustic waves, dust ion-acoustic waves etc. in quantum plasma have been studied by many researchers. Quantum effects in plasmas are usually studied with the help of two well-known formulations, viz. the Wigner-Poisson and the Schrödinger-Poisson formulations. The Wigner-Poisson model is often used in the study of quantum kinetic behaviour of plasma. The Schrödinger-Poisson model describes the hydrodynamic behaviour of plasma particles in quantum scales. The quantum hydrodynamic (QHD) model is derived by taking
velocity space moments of the Wigner equations. The QHD model generalizes the classical fluid model for plasma with the inclusion of a quantum correction term also known as the Bohm potential. The model incorporates quantum statistical effects through the equation of state. Because of simplicity, straight forward approach and numerical efficiency, the QHD equation of state. Because of simplicity, straight forward approach and numerical efficiency, the QHD model has been widely used by several researchers\textsuperscript{13,16}. Different approaches for modeling quantum plasmas in electrostatic limit have been reviewed by Manfredi\textsuperscript{16}. The QHD model as used by most researchers, is valid for quantum plasmas in the ultra-cold limit. But in many astrophysical objects, the plasma temperature is finite and not approaching zero. Recently, Eliasson and Shukla\textsuperscript{13} have developed non-linear fluid equations taking into account the moments of the Wigner equation and by using the Fermi Dirac equilibrium distribution for electrons at an arbitrary temperature. The model thus developed is expected to describe a finite temperature quantum plasma. The linear and non-linear properties of electron plasma waves in a quantum plasma have been studied by a few researchers by using QHD model\textsuperscript{13,18} valid in the ultra-cold limit. The linear and non-linear properties of electron plasma waves in a finite temperature quantum plasma including relativistic effects have been studied in the present paper.

2 Finite Temperature Model

Based on the 3D equilibrium Fermi-dirac distribution for electrons at an arbitrary temperature, Eliasson and Shukla\textsuperscript{13} derived a set of fluid equations which are valid both in the large and low temperature limits. When a plane longitudinal electron plasma wave propagates in a collisionless quantum plasma, it leads to adiabatic compression thereby causing a temperature anisotropy in the electron distribution. In quantum plasma, the classical compressibility of the electron phase fluid is violated due to quantum mechanical tunnelling. However, to a first approximation it can be assumed that the electron phase fluid is incompressible. Further, the chemical potential ($\mu$) is assumed to remain constant during the non-equilibrium dynamics of the plasma. Under these assumptions, the non-equilibrium particle density is given by:

$$n_e = \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int E^{3/2} dE \exp \left( \frac{E - \mu}{k_B T_e} + 1 \right)$$

$$= - \frac{1}{2\pi^2 \beta^{3/2}} \left( \frac{2m}{\hbar^2} \right)^{3/2} \Gamma \left( \frac{3}{2} \right) \operatorname{Li}_{3/2} \left[ - \exp (\beta \mu) \right]$$

Li\textsubscript{3/2}(y) is the polylogarithm function. In the ultra cold limit i.e $T \to 0$, we have $\beta \to \infty$ and $\mu \to E_F$.

The temperature anisotropy is given by:

$$\eta_e(x,t) = T_{0e}/T_e(x,t) = [n_0^2/n_e(x,t)]$$

The Fermi energy is given by:

$$E_F = \left( 3\pi^2 n_e \right)^{1/3} \left( \hbar^2 / 2m \right)$$

Now using the zeroth and first moments of the Wigner equation with the Fermi-Dirac distribution function and assuming that the Bohm potential is independent of the thermal fluctuations in a finite temperature plasma, one can derive the continuity and momentum equation in the following form:

$$\frac{dn_e}{dt} + \frac{\partial (n_e u_e)}{\partial x} = 0$$

$$\left( \frac{\partial}{\partial t} + u_e \frac{\partial}{\partial x} \right) (u_e \gamma) = \frac{e}{m} \frac{\partial \phi}{\partial x} - \frac{n_e V_e^2}{n_e} \frac{\partial (n_e / n_e)}{\partial x}$$

$$+ \frac{\hbar^2}{2m \gamma} \frac{\partial}{\partial x} \left[ \frac{1}{\sqrt{n_e}} \frac{\partial^2}{\partial x^2} \right]$$

where $n_e$ and $\nu_e$ are the particle density and fluid velocity of electron, respectively; $\phi$ is the electrostatic wave potential and $\nu_e = \sqrt{k_B T_e / m_e}$ is the themal speed. $G$ is the ratio of two polylogarithm functions given by:

$$G = \frac{\operatorname{Li}_{3/2}(\exp[\beta \mu])}{\operatorname{Li}_{3/2}(\exp[\beta \mu])}$$

The system is closed by the Poisson equation:

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e (n_e - n_i)$$

We now introduce the following normalization:

$$x \to x \omega_{pe} / V_{pe}, t \to t \omega_{pe}, \phi \to e \phi / 2k_BT_{pe}, n_j \to n_j / n_0$$

and $u_j \to u_j / V_{pe}$.

Here $\omega_{pe} = \sqrt{4\pi n_e e^2 / m_e}$ the electron plasma oscillation frequency and $\nu_{pe} = \sqrt{2k_BT_{pe} / m_e}$ is the Fermi speed of electrons. Using the above normalization, Eqs (3)-(5) can be recast as:
\[
\frac{\partial n_e}{\partial t} + \frac{\partial (n_e u_{ex})}{\partial x} = 0
\]
\[
\left( \frac{\partial}{\partial t} + u_{ex} \frac{\partial}{\partial x} \right) u_{ex} = \frac{\partial \phi}{\partial x} - 3G \alpha^2 n_e \frac{\partial n_e}{\partial x}
\]
\[
+ \frac{H^2}{2} \frac{\partial}{\partial x} \left[ \frac{1}{n_e} \frac{\partial^2 \sqrt{n_e}}{\partial x^2} \right]
\]
\[
\frac{\partial^2 \phi}{\partial x^2} = (n_e - n_i)
\]
where \( H = \hbar \omega_{pe} / 2 k_B T_{Fe} \) is a non-dimensional quantum parameter proportional to the quantum diffraction and \( \alpha = (V_{Fe}/V_{Fe}) \). The parameter \( H \) is proportional to the ratio between the plasma energy \( \hbar \omega_{pe} \) (energy of an elementary excitation associated with an electron plasma wave) and the Fermi energy \( k_B T_{Fe} \).

As the degeneracy parameter \( G \) determines the transition from ultracold to thermal cases it is important to have a knowledge about its order of magnitude. For example in case of Jupiter (density \( \sim 10^{12} \) m\(^{-3}\), temperature \( \sim 10^4 \) K, \( G \sim 1.4 \)); for white dwarf (density \( \sim 10^{35} \) m\(^{-3}\), temperature \( \sim 10^9 \) K, \( G \sim 4 \)).

### 3 Derivation of the Non-linear Schrödinger Equation

For high frequency electron plasma waves, we can assume that the heavier ions are immobile. Now following standard procedure, we make Fourier expansions for the field variables:

\[
\begin{bmatrix}
    n_e \\
    u_{ex} \\
    \phi
\end{bmatrix}
= \begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix}
+ \epsilon^2 \begin{bmatrix}
    n_{io} \\
    u_{io} \\
    \phi_i
\end{bmatrix}
+ \sum_{s=1}^{s=1} \epsilon_{i=1}^{i=N} \begin{bmatrix}
    n_{es} \\
    u_{es} \\
    \phi_s
\end{bmatrix} e^{i \omega t} + \begin{bmatrix}
    n_{es}^* \\
    u_{es}^* \\
    \phi_s^*
\end{bmatrix} e^{-i \omega t}
\]

\[
\begin{bmatrix}
    n_i \\
    u_i
\end{bmatrix}
= \begin{bmatrix}
    1 \\
    0
\end{bmatrix}
+ \epsilon^2 \begin{bmatrix}
    n_{io} \\
    u_{io}
\end{bmatrix}
\]

In which \( \psi = kx - \epsilon \omega t \) (\( \omega, k \) being the normalized wave frequency and wave number, respectively), the field quantities \( n_{es}, u_{es}, \phi_s, n_{es}, u_{es}, \phi_s, n_{io} \) and \( v_{io} \) are assumed to vary slowly with \( x \) and \( t \). i.e. they are supposed to be functions of \( \xi = \epsilon (x - c_\epsilon t) \) and \( \tau = \epsilon^2 t \), where \( \epsilon \) is a smallness parameter and \( c_\epsilon \) is the normalized group velocity.

Now substituting the expansion given in Eq. (7) in Eqs (6) and then equating from both sides the coefficients of exp \( (i \psi) \), exp \( (i2 \psi) \) and terms independent of \( \psi \) we obtain three sets of equations which we call I, II and III. To solve these three sets of equations we make the following perturbation expansion for the field quantities \( n_{es}, u_{es}, \phi_s, n_{es}, u_{es}, \phi_s, n_{io} \) and \( u_{io} \) which we denote by \( A \):

\[
A \approx A^{(1)} + \epsilon A^{(2)} + \epsilon^2 A^{(3)} + \ldots
\]

Solving the lowest order equations obtained from the set of equations I after substituting the expansion given in Eq. (8), we get:

\[
n^{(1)}_{es} = -k^2 \phi^{(1)}_i, u^{(1)}_{es} = - (\omega - ku_0) k \phi^{(1)}_i
\]

And the linear dispersion relation:

\[
(\omega - ku_0)^2 \gamma_s = 1 + 3G \alpha^2 k^2 + \frac{H^2 k^4 \gamma_s}{4}
\]

where \( \gamma_s = 1 + \frac{3u_0^2}{2c^2} \) and \( \gamma_s = 1 - \frac{u_0^2}{2c^2} \).

In the dimensional form, the dispersion relation becomes:

\[
(\omega - ku_0)^2 \gamma_s = \omega_{pe}^2 + 3Gk^2 V_{Fe}^2 + \frac{k^4 V_{Fe}^4 H^2 \gamma_s}{4 \omega_{pe}^2}
\]

The degeneracy parameter \( G \) determines the transition between the ultra cold and thermal cases. In the low temperature limit \( \beta \mu \rightarrow -\infty, \mu \approx E_{Fe} = (m V_{Fe}^2)/2 \) and \( G \approx 2 \beta E_{Fe}/5 \), then the dispersion relation given in Eq. (11) takes the form:

\[
(\omega - ku_0)^2 \gamma_s = \omega_{pe}^2 + \frac{3}{5} k^2 V_{Fe}^2 + \frac{k^4 V_{Fe}^4 H^2 \gamma_s}{4 \omega_{pe}^2}
\]

which is similar to the dispersion relation for electron plasma waves in a quantum plasma obtained by using one dimensional QHD model. In the high temperature limit \( \beta \mu \rightarrow -\infty \) so that \( G \rightarrow 1 \) and then the dispersion relation given in Eq. (11) reduces to:

\[
(\omega - ku_0)^2 \gamma_s = \omega_{pe}^2 + 3k^2 V_{Fe}^2 + \frac{k^4 V_{Fe}^4 H^2 \gamma_s}{4 \omega_{pe}^2}
\]

In the high temperature limit, the last term on the RHS may be neglected and then one gets the well known Bohm-Gross dispersion relation of electron plasma waves in a hot plasma. Fig. 1(a-b) shows the dependence of linear dispersion characteristics on the electron degeneracy parameter \( G \) and streaming velocity \( u_0 \). It is clear that both the electron...
The degeneracy parameter and streaming velocity increase the slope of the dispersion curve, i.e., as the values of $G$ and $u_0$ increase, the wave frequency increases for a given value of $k$.

The group velocity $c_g = d\omega / dk$ is obtained from the dispersion relation given in Eq. (13) as:

$$
c_g = \sqrt{\frac{3G\alpha^2 k + \frac{H^2 k^4 \gamma_3}{4}}{\gamma_1 \left(1 + 3G\alpha^2 k^2 + \frac{H^2 k^4}{4}\right)^{1/2}}} + u_0
$$

... (14)

Numerical analysis shows that the electron degeneracy parameter $G$ has very little effect on $c_g$ excepting at very low $k$-regions. On the other hand, the value of $c_g$ increases significantly with the streaming velocity $u_0$ (Fig. 2).

We now obtain the second harmonic terms in the lowest order equation which are obtained from the set of equations II by substituting the perturbation expansion (8).

$$
\phi_0^{(1)} = -b_2 \phi_1^{(1/2)}
$$

$$
n_{e2}^{(1)} = 4k^2 b_0 \phi_1^{(1/2)}
$$

$$
u_{e2}^{(1)} = (\omega - ku_0) \left(4b_0 k - k^3\right) \phi_1^{(1/2)}
$$

... (15)

$$
b_2 = \frac{3(\omega - ku_0)^2 k^3 \gamma_3 + 3G\alpha^2 k^5 - \frac{\gamma_3 H^2 k^7}{4}}{8(\omega - ku_0)^3 \gamma_1^2 - \frac{H^2 k^6 (\omega - ku_0) \gamma_1}{4}}
$$

... (16)

The zeroth harmonic quantities are obtained from the solutions of the lowest order equations obtained from the set of equations III after substituting the perturbation expansion given in Eq. (8).

$$
\phi_0^{(1)} = b_0 \phi_1^{(1/2)}
$$

$$
n_{e0}^{(1)} = n_{e0}^{(0)} = b_0 \phi_1^{(1/2)}
$$

$$
u_{e0}^{(1)} = \left(b_1 (c_g - u_0) - 2(\omega - ku_0) k^3\right) \phi_1^{(1/2)}
$$

... (17)

$$
u_{e0}^{(1)} = b_1 (c_g - u_0) \phi_1^{(1/2)}
$$

... (17)
The group dispersion coefficient:

\[ Q = \frac{q_i + q_z + q_z}{2 \gamma_i (\omega - ku_0) k^2} \]  \hspace{1cm} (22)

where

\[ Q = \frac{q_i + q_z + q_z}{2 \gamma_i (\omega - ku_0) k^2} \]  \hspace{1cm} (22)

and the non-linear coefficient

\[ \gamma_i = \frac{d^2}{d^2} + P \beta \frac{\partial^2}{\partial^2} = Q \alpha^2 \alpha^* \]  \hspace{1cm} (20)

where \( \alpha = \phi^{(1)} \)

The group dispersion coefficient:

\[ P = \frac{1}{2} \frac{dc_x}{dk} = \frac{1}{2 \gamma_i (\omega - ku_0)} \times \left[ 3G\alpha^2 - \gamma_i (\epsilon - u_0)^2 + \frac{3}{2} \gamma_i \kappa H^2 k^2 \right] \]  \hspace{1cm} (21)

and the non-linear coefficient

The first harmonic quantities in the second order are obtained from the solutions given in Eq. (9) by replacing \(-i\omega\) by \(-i\omega - \epsilon \frac{\partial}{\partial z} + \epsilon \omega \frac{\partial}{\partial \tau}\) and ik by

\[ \left( ik + \epsilon \frac{\partial}{\partial z} \right) \]  \hspace{1cm} (19)

and then picking out order \( \epsilon \) terms. Thus, we obtain:

\[ \phi^{(2)} = 0 \]

\[ n^{(2)} = 2ik \frac{\partial \phi^{(1)}}{\partial z} \]  \hspace{1cm} (19)

\[ u^{(2)} = (i \omega + kc - 2ku_0) \frac{\partial \phi^{(1)}}{\partial \tau} \]  \hspace{1cm} (19)

Now collecting coefficients of \( \epsilon^3 \) from both sides of the sets of equations I after substituting the perturbation expansion given in Eq. (8), we get a set of equations for the first harmonic quantities in the third order. Using the above solutions and after proper elimination, we obtain the following desired non-linear Schrödinger equation (NLSE) describing the non-linear evolution of the wave amplitude.

4 Modulational Instability

The non-linear Schrödinger Eq. (20) describes the amplitude modulation of electron plasma waves in quantum plasma including finite temperature and relativistic effects. Eq. (20) has been studied in details by many researchers in connection with the non-linear propagation of different wave modes in plasma. It is found that a uniform wave train becomes modulatorially stable when \( PQ > 0 \) and unstable when \( PQ < 0 \). The growth rate of such modulational instability attains a maximum value, \( g_{\alpha} = |Q|/P^{1/2} |\alpha|^2 \) corresponding to the wave number \( l_m = |Q|/P^{1/2} |\alpha| \) of the modulation. Thus, the instability condition depends on the sign of the product \( PQ \). Numerical computation of \( P \) and \( Q \) by using the expressions given in Eqs (21) and (22) for different values of \( k \) in terms of the system parameters \( G, H \) and \( u_0 \) has been carried out. The modulated wave is found to be unstable for all values of \( k \) except for a small region in the low \( k \) regime. It is found that the growth rate of instability increases with increase in the quantum diffraction parameter \( H \) (Fig. 3). The result is found to be in agreement with that obtained by earlier researchers. The effect of the streaming velocity \( u_0 \) and the electron degeneracy parameter \( G \) is to suppress the instability giving lower growth rate of instability (Figs 4 and 5). In the wave number region
where \( PQ < 0 \) a long-wave disturbance to the wave mode will grow. If the self-compression of the wave due to non-linearity is balanced with dispersion at certain stage of evolution a stable non-linear structure (envelope soliton) is formed; if not balanced the wave will collapse until it is absorbed by the plasma particles or converted into other waves.

The ratio \( |P/Q| \) determines the spatial extension of the solitary structure for a given maximum amplitude and vice versa in an inverse proportional manner.

Thus, it is important to study the dependence of \( P/Q \) on different plasma parameters. The variations of \( P/Q \) with respect to \( k \) for different values of \( H \) (with fixed \( u_0 \), \( G \)), for different \( u_0 \) (with fixed \( H \), \( G \)) and for different \( G \) (with fixed \( H \), \( u_0 \)) are shown in Figs 6-8, respectively. An examination of the Figs 6-8 shows that the effect of relativistic streaming and electron degeneracy on soliton amplitude and width is opposite to that of quantum diffraction parameter.

5 Discussion and Conclusions
Using one dimensional QHD model for quantum plasma at finite temperature, we have investigated the linear and non-linear properties of electron plasma waves including streaming motion and relativistic effects. It is shown that both the electron degeneracy parameter and streaming velocity have the effect of increasing the wave frequency. The electron degeneracy parameter \( G \) is found to have very little effect on the group velocity of the wave excepting at low-\( k \) regions. But the group velocity increases significantly with the increase in streaming motion.
In classical unbounded plasma, electron plasma waves are modulationally stable for all wavenumber. Inclusion of quantum effects through ultracold plasma model makes electron plasma waves modulationally unstable in two distinct regions of wavenumber. In a finite temperature quantum plasma, it is found that the electron degeneracy parameter and streaming velocity have significant contribution towards the formation and properties of electron plasma envelope solitary waves.

The QHD model as used in this paper includes finite temperature effects and it is shown that, unlike quantum diffraction effect which increases modulational instability growth rate, the electron degeneracy parameter reduces the growth rate of instability. We have also included relativistic effects in the present model. The growth rate of instability is shown to decrease with increase in streaming velocity. This result is consistent with that obtained by earlier researchers. Finally, we would like to point out that the investigation presented here may be helpful in the understanding of the basic features of electron plasma waves in dense and hot plasmas such as can be found in white dwarfs, neutron stars and intense laser-solid plasma experiments.

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