Vibration analysis of visco-elastic clamped circular plates subjected to thermal gradient

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An analysis of vibration of visco-elastic circular plate of variable thickness subjected to thermal gradient is presented here. The governing differential equation has been solved for free vibrations of visco-elastic circular plate, which is clamped along the boundary. Galerkin’s technique has been applied to obtain corresponding natural frequencies in the form of explicit formulae. Deflection, time period and logarithmic decrement at different points for the first two modes of vibration are calculated for various types of thermal gradient and taper constant and are illustrated with tables and graphs.

Sufficiently high temperatures are encountered in various engineering branches such as nuclear, power generation, aeronautical, chemical, etc., where metals and their alloys exhibit visco-elastic behavior. For such conditions, vibration analyses has to be carried out using constitutive equations of visco-elastic theory rather than those of elastic theory. Most of the investigations on vibration have been carried out using theory of elasticity. However, references are available on the vibrations of uniform visco-elastic isotropic beams and plates. Sobotka has considered free vibrations of uniform visco-elastic orthotropic rectangular plates. Muki and Sternberg have investigated the stress analysis of visco-elastic plates at elevated temperatures.

The present investigation is aimed at a parametric study of the vibration of visco-elastic circular plate of variable thickness subjected to thermal gradient. The assumptions of small deflection and linear, isotropic visco-elastic properties are made. It is further assumed that the visco-elastic properties of the plate are of the Kelvin type. Numerical calculations have been done using the material constants of the Duralium alloy.

Analysis

The equation of motion of a visco-elastic isotropic plate of variable thickness may be written in the form:

\[ M_{xx} + 2M_{x,y} + M_{y,y} = \rho \ddot{w}_{xy} \]  

(1)

A comma followed by a suffix denotes partial differentiation with respect to that variable. The expressions for \( M_x, M_y \) and \( M_{xy} \) are given by:

\[
M_x = -\tilde{D} D_1 (w_{xx} + \nu w_{yy}) \\
M_y = -\tilde{D} D_1 (w_{yy} + \nu w_{xx}) \\
M_{xy} = -\tilde{D} D_1 (1 - \nu) w_{xy} 
\]

(2)

On substituting the values of \( M_x, M_y \) and \( M_{xy} \) from Eq. (2) in Eq. (1), we get:

\[
\tilde{D}(D_1(w_{xxx} + 2w_{xxy} + w_{yyy}) + 2D_{1,s} (w_{x,s} + w_{y,s}) + 2D_{1,s} (w_{y,s} + w_{x,s}) \\
+ D_{1,t} (w_{x,t} + \nu w_{y,t}) + D_{1,t} (w_{y,t} + \nu w_{x,t}) \\
ox 2(1 - \nu) D_{1,t} w_{xy} + \rho \ddot{w}_{xy} = 0 
\]

(3)

The solution of Eq. (3) can be sought in the form of products of two functions as:

\[ w(x, y, t) = w(x, y) T(t) \]

(4)

Using Eq. (4) in Eq. (3) and simplifying, one gets:

\[
\{D_1(w_{xxx} + 2w_{xxy} + w_{yyy}) + 2D_{1,s} (w_{x,s} + w_{y,s}) + 2D_{1,s} (w_{y,s} + w_{x,s}) \\
+ D_{1,t} (w_{x,t} + \nu w_{y,t}) + D_{1,t} (w_{y,t} + \nu w_{x,t}) \\
+ 2(1 - \nu) D_{1,t} w_{xy} \} / \rho \ddot{w} = -\tilde{T} / \tilde{D} 
\]

(5)
Relation (5) is satisfied if both its sides are equal to a constant. Denoting this constant by \( p^2 \), we get:

\[
D_t (w_{xx} + 2w_{xy} + w_{yy}) + 2D_{tt} (w_{xx} + w_{yy}) + 2D_{t,tt} (w_{xx} + v w_{yy}) + 2(1-v)D_{t,tt} w_{tt} - p^2 w = 0
\]  
\[
... \quad (6)
\]

and,

\[
\bar{T} + p^2 \bar{D}T = 0 \quad ... \quad (7)
\]

Eq. (6) is a differential equation of motion for elastic isotropic plate of variable thickness and Eq. (7) is a differential equation of time functions of free vibrations of visco-elastic plate. It is assumed that the visco-elastic isotropic circular plate is subjected to a steady two dimensional temperature distribution given by:

\[
\tau = \tau_0 (1-X) \quad ... \quad (8)
\]

where, \( \tau \) denotes the temperature excess above the reference temperature at any point on concentric circles and \( \tau_0 \) denotes the temperature excess above the reference temperature at the center of the circular plate \((x = y = 0)\).

Solution of Equation (6)

The temperature dependence of the modulus of elasticity, for most engineering materials, is given by a relation of the type:

\[
E(\tau) = E_0 (1 - \gamma \tau) \quad ... \quad (9)
\]

where, \( E_0 \) is value of the modulus at some reference temperature and \( \gamma \) is the slope of the variation of \( E \) with \( \tau \). With the reference temperature taken as the temperature on the boundary, the modulus variations, in view of the expressions (8) and (9), become:

\[
E = E_0 [1 - \alpha (1 - X)] \quad ... \quad (10)
\]

where \( \alpha = \gamma \tau_0 \) \((0 \leq \alpha \leq 1)\).

The thickness variation of the plate is assumed to be of the form:

\[
h = h_0 (1 - \beta X) \quad ... \quad (11)
\]

where \( h_0 \) is the thickness of the plate at the center \((x = y = 0)\). The flexural rigidity of the plate can now be written as:

\[
D_t = \frac{E_0 [1 - \alpha (1 - X)] h_0^3 (1 - \beta X)^5}{12 (1 - v^2)} \quad ... \quad (12)
\]

Here the Poisson’s ratio is assumed constant. Using Eq. (11) and Eq. (12) in Eq. (6), one gets:

\[
B_t (w_{xx} + 2w_{xy} + w_{yy}) + 2B_{tt} (w_{xx} + w_{yy}) + B_{t,tt} (w_{xx} + v w_{yy}) + 2B_{t,tt} w_{tt} - \left( \frac{p^2 l}{a^4} \right) w = 0 \quad ... \quad (13)
\]

where,

\[
B_1 = [1 - \alpha (1 - X)] [1 - \beta X]^2
\]

\[
B_2 = \frac{2X}{a^2} [1 - \beta X] [\alpha (1 - \beta X) - 3\beta (1 - \alpha (1 - X))]
\]

\[
B_3 = B_2 \frac{y}{X}
\]

\[
B_4 = 2 \alpha^2 [(1 - \beta) (\alpha (1 - \beta X) - 12\alpha \beta^2 / a^2 - 3\beta (1 - \alpha (1 - X))] + 12 \beta X^2 / a^2 [1 - \alpha (1 - X)]
\]

\[
B_5 = 2 \alpha^2 [(1 - \beta X) (\alpha (1 - \beta X) - 12\alpha \beta^2 / a^2 - 3\beta (1 - \alpha (1 - X))] + 12 \beta X^2 / a^2 [1 - \alpha (1 - X)]
\]

\[
B_6 = 24 \beta XY / a^2 [\alpha (1 - \beta X) + 1 - \alpha (1 - X)]
\]

and,

\[
l = 12 (1 - v^2) \rho a - \frac{p^2 l}{h_0 E_0} \] where \( a^2 \) is a frequency parameter.

Free vibration of clamped circular plate

The deflection function \( W(x, y) \) of the plate is assumed to be a finite sum of characteristic function \( W_k (x, y) \)

\[
W(x, y) = \sum_{k=1}^{K} C_k W_k (x, y) \quad ... \quad (14)
\]
For a clamped plate, boundary conditions are that the deflection $W = 0$ and the slope $W_{x} = W_{y} = 0$ along $1-X=0$. Using Galerkin's technique, it is required that:

$$
\int_{A} L(W(x,y))W(x,y)dx\,dy = 0 
$$

where $L[W(x,y)]$ is the left hand side of Eq. (13). Taking the first two terms of the sum Eq. (14), for the function $W$ as a solution of Eq. (13), one has:

$$
W = C_{1}(1-X)^{2} + C_{2}(1-X)^{3} 
$$

where $C_{1}$ and $C_{2}$ are undetermined coefficients.

Substituting relation (16) into Eq. (15), and then eliminating $C_{1}$ and $C_{2}$, we get the frequency equation as:

$$
\begin{bmatrix} F_{1} & F_{2} \\ F_{2} & F_{1} \end{bmatrix} = 0 
$$

where,

$$
F_{1} = \frac{2}{a^{2}} \left[ \left( \frac{16}{3} - 2\alpha - \frac{26}{3} \beta + \frac{59}{15} \beta^{2} + \frac{11}{5} \alpha \beta + \frac{11}{15} \alpha \beta^{2} \right) 
- \left( \frac{2}{5} \alpha + \frac{26}{5} \beta + \frac{7}{15} \alpha \beta + \frac{3}{5} \alpha \beta^{2} \right) - p^{2} \times \frac{1}{20} \right] 
$$

$$
F_{2} = \frac{1}{a^{2}} \left[ \left( \frac{8}{5} \alpha - \frac{54}{5} \beta + \frac{67}{15} \beta^{2} + \frac{13}{3} \alpha \beta - \frac{823}{840} \alpha \beta^{2} \right) 
- \left( \frac{6}{5} \alpha + \frac{18}{5} \beta - \frac{11}{15} \alpha \beta - \frac{19}{15} \alpha \beta^{2} \right) - p^{2} \times \frac{1}{12} \right] 
$$

$$
F_{3} = \frac{2}{a^{2}} \left[ \left( \frac{24}{5} - 3\alpha - \frac{27}{5} \beta + \frac{67}{35} \beta^{2} + \frac{98}{35} \alpha \beta - \frac{177}{280} \alpha \beta^{2} \right) 
+ \left( \frac{3}{5} \alpha + \frac{9}{5} \beta - \frac{33}{35} \alpha \beta + \frac{148}{280} \alpha \beta^{2} \right) - p^{2} \times \frac{1}{28} \right] 
$$

The frequency Eq. (17) is a quadratic equation in $p^{2}$ from which the values of $p^{2}$ can be found.

Thus, deflection function $w(x, y)$ can be obtained from Eq. (16) after determining $C_{1}$ and $C_{2}$ from Eq. (17). Choosing $C_{1} = 1$, one obtains $C_{2} = -F_{1}/F_{2}$ from first Eq. of (17). Therefore, $w(x, y)$ becomes:

$$
W(x, y) = (1-X)^{2} + \left( \frac{F_{1}}{F_{2}} \right) (1-X)^{3} 
$$

Time functions of vibrations of visco-elastic plates

Time functions of free vibrations of visco-elastic plates are defined by the general ordinary differential Eq. (7). Their form depends on the visco-elastic operator $\tilde{D}$. For Kelvin's model one can have:

$$
\tilde{D} = \left( 1 + \frac{\eta_{0}}{G \cdot d} \right) 
$$

Taking temperature dependence of shear modulus and visco-elastic constant in the same form as that of Young's modulus, we have:

$$
G(\tau) = G_{0} \cdot (1 - \gamma_{1} \tau) 
$$

$$
\eta(\tau) = \eta_{0} \cdot (1 - \gamma_{2} \tau) 
$$

where $G_{0}$ is shear modulus and $\eta_{0}$ is visco-elastic constant at some reference temperature i.e. at $\tau = 0$. $\gamma_{1}$ and $\gamma_{2}$ are slope variation of $\tau$ with $G$ and $\eta$ respectively. Using Eq. (8) in relation (20), one can have:

$$
G = G_{0} \cdot (1 - \alpha_{1} \cdot (1-X)) 
$$

$$
\eta = \eta_{0} \cdot (1 - \alpha_{2} \cdot (1-X)) 
$$

where $\alpha = \gamma_{1} \tau_{0} (0 \leq \alpha_{1} \leq 1)$ and $\alpha_{2} = \gamma_{2} \tau_{0} (0 \leq \alpha_{2} \leq 1)$.

Using Eq. (21) in relation (19), we get:

$$
\tilde{D} = 1 + q \frac{d}{dt} 
$$

where,

$$
q = \frac{\eta_{0} \cdot (1 - \alpha_{2} \cdot (1-X))}{G_{0} \cdot (1 - \alpha_{1} \cdot (1-X))} 
$$

Using Eq. (22) in Eq. (7), one obtains:

$$
\ddot{T} + p^{2} q \dot{T} + p^{2} T = 0 
$$

Eq. (24) is a differential equation of second order for time function $T$. The time period of the vibration of the plate is given by:
\[ K = \frac{2\pi}{p} \] ... (25)

where \( p \) is frequency given by Eq. (17). The logarithmic decrement of the vibration is given by the standard formulae:

\[ \lambda = \log \left( \frac{w_2}{w_1} \right) \] ... (26)

where \( w_1 \) is the deflection at any point of the plate at a time period \( K = K_1 \) and \( w_2 \) is the deflection at the same point at the time period succeeding \( K_1 \).

**Results and Discussion**

Logarithmic decrement, time period and deflection are computed for a clamped visco-elastic circular plate for different values of temperature constants \( \alpha, \alpha_1, \alpha_2 \); taper constant \( \beta \), and the distance from the center \( X \). Results are presented through both graphs and tables. For numerical computation, following material properties as reported for Duranium\(^3\), are used:

\[ E_0 = 7.08 \times 10^6 \text{ N/m}^2, G_0 = 2.632 \times 10^{10} \text{ N/m}^2 \]
\[ \eta_0 = 14.612 \times 10^5 \text{ N.S/m}^2, \rho = 2.80 \times 10^3 \text{ kg/m}^3, \]
\[ \nu = 0.345 \]

The thickness of the plate at the center is taken as \( h_0 = 0.01 \text{ m} \). To study the effect of taper in plate, time period \( K \) and logarithmic decrement \( \beta \) have been computed for different values of \( \beta \) keeping \( \alpha, \alpha_1, \alpha_2 \) zero. In other words, the plate is considered to be at a constant temperature. The variation of time periods and logarithmic decrement respectively for both the first and the second mode of vibrations versus \( \beta \), the taper constant is shown in Figs 1 and 2. It is interesting to see that while the effect of increasing taper is more prominent for the time period in the first mode, on logarithmic decrement this effect is more prominent in the second mode. It has also been found that logarithmic decrement is independent of \( X \) in this case.

Effect of taper has also been studied for the case where there is temperature gradient. The values of temperature constants \( \alpha, \alpha_1, \alpha_2 \) were assumed to be 0.2, 0.3, 0.6 respectively, and logarithmic decrement \( \lambda \), time period \( K \) and deflection \( w \) were calculated for the first two modes of vibrations for \( \beta = 0.0 \) (uniform thickness) and \( \beta = 0.6 \) (varying thickness) at different values of \( X \) and are shown in Tables 1a-1c. It can be seen from the Tables that logarithmic decrement and time period increases and deflection decreases for the plate with taper for both the modes of vibration as compared to the plate with uniform thickness.

To study the effect of temperature gradient, \( \lambda, K \) and \( w \) are calculated and compared in Tables 2a-2c for \( \beta = 0.6 \) for the following two cases at different values of \( X \). (Case I: \( \alpha = \alpha_1 = \alpha_2 = 0.0 \); Case II: \( \alpha = 0.2, \alpha_1 = 0.3, \alpha_2 = 0.6 \)). The effect of thermal gradient on

![Fig. 1—Time period K versus taper constant β](#)

![Fig. 2—Logarithmic decrement versus taper constant β](#)
logarithmic decrement, time period and deflection is presented in Tables 2a-2c. It is interesting to observe from Table 2a that the logarithmic decrement is constant across the plate for constant temperature case. For the plate with temperature gradient, the logarithmic decrement increases as compared to the plate at constant temperature for all values of X. Table 2b shows that the effect on period K is much smaller, in fact for the second mode the effect was so small that the difference in the values occurred at the fourth place of decimal. Table 2c shows that the deflection increases for a plate with temperature gradient compared to a plate at constant temperature at all values of X.

Conclusions

On the basis of these limited analytical investigations, it is found that: (i) The effect of increasing taper is more prominent for the time period in the first mode, on logarithmic decrement this effect is more prominent in the second mode; (ii) The logarithmic decrement and time period increases and deflection decreases for the plate with taper for both the modes of vibration as compared to the plate with uniform thickness; (iii) The logarithmic decrement is constant across the plate for constant temperature case; (iv) For the plate with temperature gradient, the logarithmic increment increases as compared to the plate at constant temperature; and, (v) The deflection increases for a plate with temperature gradient compared to a plate at constant temperature.

References

9 Sobotka Zdenek, Acta Techn CSAV, No.6 (1978) 678-705.