Elegant and accurate closed form solutions to predict vibration and buckling behaviour of slender beams on Pasternak foundation

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Elegant and accurate closed form solutions to predict the vibration and buckling of slender beams on Pasternak (two parameter elastic) foundation have been obtained using simple single term trigonometric functions which satisfy the geometric boundary conditions in conjunction with the Rayleigh-Ritz method. Bernoulli-Euler beam theory is used for the analysis. Simply supported, cantilever and clamped boundary conditions are considered in the present study. Numerical results for the fundamental frequency and buckling load parameters are presented in the form of Tables. The present results show an excellent agreement when compared with those obtained by the versatile finite element method. An anomaly observed in the case of the buckling problem for very high first foundation stiffness parameter, is explained clearly for the case of a simply supported beam and corrective measure in terms of the mode of buckling is provided.

The vibration and buckling analysis of slender beams, which can be treated by Bernoulli-Euler beam theory on elastic foundation, is of importance in many fields of engineering. In general, the Winkler type of elastic foundation is widely used by engineers because of the simplicity of the model. The Winkler type of foundation acts as if it consists of an infinitely large number of closely spaced elastic springs. However, the Winkler hypothesis neglects the interaction between the adjacent springs and hence cannot represent the characteristics of many practical foundations. Formulations with two parameter elastic foundation have been suggested by many investigators to study these problems and give more insight to research engineers on this topic. These are less restrictive than the Winkler foundation and take care of the realistic situation more accurately. A brief review of the various types of two parameter foundations is given by Zhaohua and Cook, which can be used as a ready reference to have an idea of two parameter elastic foundations and who have shown that mathematically the two parameter foundations can be represented by:

\[ p(x) = k_1 w(x) - k_2 \frac{d^2 w(x)}{dx^2} \]  

where \( p(x) \) is the pressure exerted by the foundation on the beam for a lateral displacement \( w(x) \), \( x \) being the axial coordinate, \( k_1 \) is the first foundation stiffness (or Winkler stiffness) and \( k_2 \) is the second foundation stiffness with different physical significance particular to the type of the two parameter foundation. The most commonly used two parameter foundation is the Pasternak foundation model.

Free vibration and buckling analysis of beams can be seen in the works of Franciosi and Masi and Naidu and Rao. While the finite element method (FEM) is a versatile tool to solve the complex problems discussed above, elegant and accurate closed form solutions for the vibration and buckling problems of beams of various boundary conditions are very attractive for design engineers. An attempt is made in this paper to provide such closed form solutions for uniform beams on the widely used, two parameter Pasternak foundation using one term admissible functions satisfying the geometric boundary conditions, for the first time in literature.

**Formulation**

For a beam of length \( L \) on Pasternak foundation (Fig. 1) executing harmonic oscillations, the total potential energy \( \pi_v \) is given by:

\[ \pi_v = \frac{EI}{2} \int_0^L (w''')^2 \, dx + k_1 \int_0^L w'^2 \, dx + k_2 \int_0^L (w')^2 \, dx - \frac{m \omega^2}{2} \int_0^L w^2 \, dx \]  

where, \( E \) is the Young's modulus, \( I \) is the area moment of inertia, \( m \) is the mass per unit length, \( w \) is the transverse displacement and \( \omega \) is the radian frequency.
Considering a simply supported beam, for the first mode of vibration $w$ is taken as:

$$w = a \sin \frac{\pi x}{L}$$

...(3)

where, $a$ is an undetermined coefficient. This function for $w$ satisfies all the geometric boundary conditions of a simply supported beam.

Substituting Eq. (3) in Eq. (1), and minimising the Lagrangian, using the Rayleigh-Ritz method, with respect to $a$, we get after simplification:

$$\lambda_f = \pi^4 + \lambda_{F1} + \lambda_{F2} \pi^4$$

...(4)

where, $\lambda_f$ is the fundamental frequency parameter ($= m \omega^2 L^4 / EI$). In Eq. (4),

$$\lambda_{F1} = \frac{k_1 L^4}{EI}$$

...(5)

and

$$\lambda_{F2} = \frac{k_2 L^2}{\pi^2 EI}$$

...(6)

are the first stiffness (Winkler) and second stiffness parameters respectively.

Similarly for the buckling problem of a beam on Pasternak foundation with end concentrated compressive load $P$, the total potential energy $\pi_b$ is given by:

$$\pi_b = \frac{EI}{2} \int_0^l (w')^2 dx + \frac{k_1}{2} \int_0^l w^2 dx$$

$$+ \frac{k_2}{2} \int_0^l (w')^2 dx - \frac{P}{2} \int_0^l (w')^2 dx$$

...(7)

For the case of a simply supported beam, following the same procedure as in the vibration problem, the buckling load parameter $\lambda_b$ ($= P L^2 / EI$) can be obtained by minimising the total potential energy using the Rayleigh-Ritz method as:

$$\lambda_b = \frac{\lambda_{F1}}{\pi^2} + (1 + \lambda_{F2}) \pi^2$$

...(8)

For cantilever and clamped beams considered in this paper, the functions for $w$ satisfying the geometric boundary conditions, respectively, are:

$$w = a \left(1 - \cos \frac{\pi x}{2L}\right)$$

...(9)

and

$$w = a \left(1 - \cos \frac{2\pi x}{L}\right)$$

...(10)

Following the procedure used for simply supported beams, using the admissible functions given in Eqs (9) and (10), the expressions for $\lambda_f$ and $\lambda_b$ are obtained as:

$$\lambda_f = \lambda_{F1} + \frac{(1 + 4 \lambda_{F2}) \pi^4}{48 - \frac{128}{\pi}}$$

...(11)

and

$$\lambda_b = \frac{\lambda_{F1}}{\pi^2 \left(12 - \frac{32}{\pi}\right)} + \left(1 + \lambda_{F2}\right) \pi^2$$

...(12)

for the case of a cantilever beam and

$$\lambda_f = \lambda_{F1} + \frac{4}{3} (4 + \lambda_{F2}) \pi^4$$

...(13)

and

$$\lambda_b = \frac{3}{4 \pi^2} \lambda_{F1} + (4 + \lambda_{F2}) \pi^2$$

...(14)

for the clamped beam.

**Results and Discussion**

Numerical results for $\lambda_f$ (fundamental frequency parameter) and $\lambda_b$ (buckling load parameter for the
first mode) are presented for simply supported beams in Tables 1 and 2, for cantilever beams in Tables 3 and 4 and for clamped beams in Tables 5 and 6 respectively for various values of \( \lambda_{11} \) and \( \lambda_{22} \). Results obtained by FEM are also included in these Tables for comparison. It may be noted here that FEM results for cantilever beam are calculated from the same FE code used by Naidu and Rao. All the FE solutions are obtained using 20 equal length elements for idealising the beam based on the convergence study and give an

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Table 1 — Frequency parameter \( \lambda_{11} \) of a simply supported beam

Table 2 — Buckling load parameter \( \lambda_{11} \) of a simply supported beam

Table 3 — Frequency parameter \( \lambda_{11} \) of a cantilever beam

Table 4 — Buckling load parameter \( \lambda_{11} \) of a cantilever beam

Table 5 — Frequency parameter \( \lambda_{11} \) of a clamped beam

Table 6 — Frequency parameter \( \lambda_{11} \) of a clamped beam
The buckling load parameter in the present case, the buckling load parameter obtained for existing critical foundation parameters and for eigenvalues in FEM captures the lowest eigenvalue corresponding to $m=3$ and the present results are obtained for $m=1$. The accuracy of at least up to four significant figures depending on the boundary conditions. The beam element used is the usual Ritz type element with two nodes and two degrees of freedom (translation and rotation) per node.

It may be noted here that the differential equations governing the present vibration and buckling problem can be exactly solvable. But the present differential equations are more complex than those of the same without the Pasternak foundation. Hence, the resulting transcendental equations (containing two foundation stiffness parameters) is more complex to solve, to obtain the vibration and buckling parameters. Further, for a given combination of the two foundation stiffness parameters, the transcendental equations, corresponding to the vibration and buckling problems, have to be solved independently for that combination. Thus, it is an enormous computational effort to obtain the whole data presented in this paper.

On the other hand, the formulas presented in this paper can be effectively and easily used to obtain the frequency and buckling parameters of slender beams with commonly used boundary configurations. The numerical results presented in this paper speak for themselves about the accuracy that can be achieved from the simple formulas.

As the assumed mode shape in the present study for simply supported beam is the exact mode, the present results and the FE results are exactly the same. However, in the case of the buckling problem, one can see that the present results for $\lambda_{s1}=10^4$ are different from FE results. This aspect is further investigated and it is found that this anomaly is due to the fact that, for a beam on elastic foundation there exist critical foundation parameters and for foundation stiffnesses above these values the lowest buckling load occurs at higher modes. It can be worked out that for this value of $\lambda_{s1}$, the minimum buckling load occurs at $m=3$, where $m$ is the mode number. The algorithm used to calculate the eigenvalues in FEM captures the lowest eigenvalue (in the present case, the buckling load parameter) corresponding to $m=3$ and the present results are obtained for $m=1$.

To demonstrate the usefulness and effectiveness of the present solution, an admissible function for $w$ covering all the modes is assumed as:

$$w = a_m \sin \frac{m \pi x}{L} \quad \ldots (15)$$

where, $a_m$ is an undetermined coefficient. The buckling load parameter $\lambda_{mn}$ for any $m$, is obtained following the procedure described in the previous section as,

$$\lambda_{mn} = \frac{\lambda_{s1}}{m^2 \pi^2} + (m^2 + \lambda_{s2}) \pi^2 \quad \ldots (16)$$

The value of $\lambda_{mn}$ for $m=3$ ($\lambda_{33}$) is given in Table 2 for $\lambda_{s1}=10^4$ and it can be seen that this value is matching exactly with the FE results (Table 2).

As there is no transition foundation stiffness $\lambda_{s1}$ for the vibration problem, the assumed mode shape of $\sin \pi x/L$ gives exactly the same results as those obtained by FEM for the simply supported beam (Table 1).

Similar values for $\lambda_{f1}$ and $\lambda_{f2}$ are given in Tables 3 and 4 for a cantilever beam and in Tables 5 and 6 for a clamped beam. The assumed trigonometric admissible functions for these cases are so accurate that a maximum error of 2% for the vibration problem (with $\lambda_{s1}=\lambda_{s2}=0$) and excellent match with FE results for the buckling problem is observed in the case of a cantilever beam (percentage error decreasing with increasing $\lambda_{s1}$ or $\lambda_{s2}$). In the case of a clamped beam the present results are in excellent agreement with those obtained by FEM. For the buckling problem, the results for $\lambda_{f1}$ are given for $\lambda_{s2}$ values up to 100. Similar to the simply supported beam case, the phenomenon of changing buckled mode shape is noticed for these boundary conditions also. But, further investigation of this phenomenon as in the case of a simply supported beam, is not possible as generalised mode shapes for a given mode number $m'$ can not be assumed.
Conclusions

Elegant and accurate closed form solutions are obtained to predict the vibration and buckling behaviour of slender beams, where Bernoulli-Euler beam theory is applicable, on Pasternak foundations. Simple one term trigonometric admissible functions are used satisfying the three types of boundary conditions and the corresponding expressions for $\lambda_j$ and $\lambda_b$ are derived using the Rayleigh-Ritz method. The numerical results obtained from the present formulation are in excellent agreement with those obtained by the FEM. The anomaly of the higher buckling load obtained by the present formulation for the case of a simply supported beam is resolved with a proper explanation and modification of the admissible function to include the mode number. The formulas given in this paper, for the first time in literature, are expected to be very useful for structural design engineers.

References

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